MATH0004 Analysis 2

Solutions to the main examination paper for the academic year 2018/2019

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Solution to question 1

(a)

Definition 1 *We say that* $L = \lim_{n\to\infty} x_n$ *if* $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ *such that*

$$n > N \qquad \Longrightarrow \qquad |x_n - L| < \varepsilon.$$

[4 marks; bookwork]

(b)

Definition 2 *A sequence of real numbers* $\{x_n\}$ *is said to be a* Cauchy sequence *if* $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ *such that*

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < \varepsilon.$$

[4 marks; bookwork]

(c)

Theorem 1 *A sequence of real numbers* $\{x_n\}$ *converges iff it is a Cauchy sequence.*

[4 marks; bookwork]

(d)(i) We have

$$\begin{aligned}
|x_m - x_n| &= |(x_{n+1} - x_n) + (x_{n+2} - x_{n+1}) + \ldots + (x_m - x_{m-1})| \\
&\le |x_{n+1} - x_n| + |x_{n+2} - x_{n+1}| + \ldots + |x_m - x_{m-1}| \\
&\le \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(m-1)m} \\
&= \left[\frac{1}{n} - \frac{1}{n+1}\right] + \left[\frac{1}{n+1} - \frac{1}{n+2}\right] + \ldots + \left[\frac{1}{m-1} - \frac{1}{m}\right] = \frac{1}{n} - \frac{1}{m} < \frac{1}{n}\,.
\end{aligned}$$

[7 marks; unseen exercise]

(d)(ii) For $m = n$ the inequality

$$|x_m - x_n| < \frac{1}{m} + \frac{1}{n} \tag{1}$$

is obvious, whereas for $m \neq n$ it follows from (d)(i) (consider two cases, $m > n$ and $m < n$). **[3 marks; unseen exercise]**

(d)(iii) Let ε be an arbitrary positive number. Choose a natural number N such that

$$N > \frac{2}{\varepsilon}\,. \tag{2}$$

Formulae (1) and (2) tell us that for $m, n > N$ we have $|x_m - x_n| < \varepsilon$, which means that our sequence is Cauchy. By Cauchy's General Principle of Convergence it converges. **[3 marks; unseen exercise]**

Solution to question 2

(a)

Definition 3

(i) A partition P *of the interval* $[a, b]$ *is a finite sequence of real numbers* $P = \{x_0, x_1, \ldots, x_n\}$ *such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The set of all partitions of a given interval $[a, b]$ *will be denoted by* $\mathcal{P}[a, b]$.

(ii) The mesh (norm, width) *of the partition* $P = \{x_0, x_1, \ldots, x_n\}$ *is the number*

$$\|P\| := \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

[3 marks; bookwork]

(b)(i)

Definition 4 *Let* $f \in \mathcal{B}[a, b]$, $P \in \mathcal{P}[a, b]$, $P = \{x_0, x_1, \ldots, x_n\}$. *The* lower Darboux sum *of* f *with respect to the partition* P *is defined as*

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1})\,,$$

where $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$.

[3 marks; bookwork]

(b)(ii)

Definition 5 *Let* $f \in \mathcal{B}[a, b]$. *The* lower Riemann integral *of* f *over* $[a, b]$ *is defined as*

$$\underline{\int_a^b} f(x)\, dx := \sup_{P \in \mathcal{P}[a,b]} L(f, P)\,.$$

[3 marks; bookwork]

(c)(i) We have $\inf\limits_{x \in [x_{i-1},\, x_i]} f(x) = x_{i-1}$, so

$$\left(\inf_{x \in [x_{i-1},\, x_i]} f(x) \right) (x_i - x_{i-1}) = x_{i-1}(x_i - x_{i-1}) = \frac{1}{2}(x_i^2 - x_{i-1}^2) - \frac{1}{2}(x_i - x_{i-1})^2. \tag{3}$$

Formula (3) implies

$$\frac{1}{2}(x_i^2 - x_{i-1}^2) - \frac{1}{2}(x_i - x_{i-1})\|P\| \le \left(\inf_{x\in[x_{i-1},\,x_i]} f(x)\right)(x_i - x_{i-1}) < \frac{1}{2}(x_i^2 - x_{i-1}^2). \tag{4}$$

[9 marks; unseen exercise]

(c)(ii) Summing up the inequalities (4) over $i = 1, 2, \ldots, n$, we get

$$\frac{1}{2}(1 - \|P\|) \le L(f, P) < \frac{1}{2}\,. \tag{5}$$

[3 marks; unseen exercise]

(d) The right inequality (5) tells us that $\frac{1}{2}$ is an upper bound for lower Darboux sums, whereas the left inequality (5) tells us that $\frac{1}{2}$ is the least upper bound for lower Darboux sums (because the mesh can be chosen to be arbitrarily small). Hence, by the definition of the lower Riemann integral (Definition 5), $\underline{\int_0^1} x\,dx = \frac{1}{2}$. **[4 marks; unseen exercise]**

Solution to question 3

(a) Suppose $\forall\varepsilon > 0\ \exists P \in \mathcal{P}[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon. \tag{6}$$

Need to prove that $f \in \mathcal{R}[a, b]$.

Let ε be an arbitrary positive number. Choose a $P \in \mathcal{P}[a, b]$ such that (6) holds. Then

$$U(f, P) - \varepsilon < L(f, P) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(f, P),$$

so

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, this implies

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0,$$

which means that $f \in \mathcal{R}[a, b]$. **[7 marks; bookwork]**

(b)

Definition 6 *We say that the function* $f : [a, b] \to \mathbb{R}$ *is* uniformly continuous *if* $\forall\varepsilon > 0\ \ \exists\delta > 0$ *such that*

$$|x - y| < \delta \quad \& \quad x, y \in [a, b] \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon.$$

[4 marks; bookwork]

(c) Consider a function $f \in C[a, b]$. Let ε be an arbitrary positive number. Then, by the fact that a continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous and Definition 6, $\exists\delta > 0$ such that

$$|x - y| < \delta \quad \& \quad x, y \in [a, b] \qquad \Longrightarrow \qquad |f(x) - f(y)| < \frac{\varepsilon}{2(b - a)}\,.$$

Choose a partition $P = \{x_0, x_1, \dots, x_n\}$ with $\|P\| < \delta$. Then

$$M_i - m_i \le \frac{\varepsilon}{2(b-a)}\,;$$

here we use the usual notation $m_i := \inf_{x\in[x_{i-1},x_i]} f(x)$, $M_i := \sup_{x\in[x_{i-1},x_i]} f(x)$. We get

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \le \frac{\varepsilon}{2(b-a)}\sum_{i=1}^{n}(x_i - x_{i-1}) = \frac{\varepsilon}{2} < \varepsilon$$

and the result from (a) tells us that $f \in \mathcal{R}[a,b]$. **[8 marks; bookwork]**

(d) Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e & \text{if} \quad x = 0, \\ \sum_{n=0}^{[x^{-1}]} \frac{1}{n!} & \text{if} \quad x > 0, \end{cases}$$

where $\left[x^{-1}\right]$ is the integer part of x^{-1}. Our function is decreasing on $[0,1]$, hence it is integrable on $[0,1]$.

Now, take an $n \in \mathbb{N}$, $n \ge 2$, and set $x_0 = \frac{1}{n}$. Then

$$\lim_{x\to x_0^-} f(x) - \lim_{x\to x_0^+} f(x) = \frac{1}{n!}\,,$$

so f is not continuous at x_0. We have established that our function has an infinite number of discontinuities. **[6 marks; unseen exercise]**

Solution to question 4

(a) Put $h(x) := f(x) - g(x)$. Then $h'(x) = 0$ for all $x \in (-1,1)$. I claim that

$$h(x) = h(0), \qquad \forall x \in (-1,1). \tag{7}$$

Case 1: $x = 0$. Obviously, (7) is true.

Case 2: $x \in (0,1)$. Applying the Mean Value Theorem to the function h on the interval $[0,x]$ we get

$$\frac{h(x) - h(0)}{x - 0} = h'(\xi) \tag{8}$$

for some $\xi \in (0,x)$. But $h'(\xi) = 0$, so (8) implies (7).

Case 3: $x \in (-1,0)$. Applying the Mean Value Theorem to the function h on the interval $[x,0]$ we get

$$\frac{h(0) - h(x)}{0 - x} = h'(\xi) \tag{9}$$

for some $\xi \in (x,0)$. But $h'(\xi) = 0$, so (9) implies (7). **[8 marks; unseen exercise, but somewhat similar to a lemma proved in the lectures and an exercise from an exercise sheet]**

(b) In the lectures the radius of convergence was defined as follows. Introduce the set

$$S := \left\{ r \ge 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ converges for } x = r \text{ or } x = -r \right\}. \tag{10}$$

Obviously, $\{0\} \subset S \subset [0, +\infty)$. The *radius of convergence* of a power series is the extended real number

$$R := \sup S$$

where S is the set (10).

Another possible definition is that the radius of convergence of a power series is the extended real number R such that for $|x| < R$ the series converges absolutely and for $|x| > R$ the series diverges. I will accept this definition even though it is not *a priori* obvious that an R with these properties exists. **[3 marks; bookwork]**

(c)

Theorem 2 *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.*

[4 marks; bookwork]

(d) Fix an $x \neq 0$ and apply the ratio test:

$$\left| \frac{\frac{(-1)^{k+1}}{2k+3} x^{2k+3}}{\frac{(-1)^k}{2k+1} x^{2k+1}} \right| = \frac{2k+1}{2k+3} x^2 = \frac{x^2}{1 + \frac{2}{2k+1}} \to x^2$$

as $k \to \infty$. Hence, the power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is 1. **[4 marks; unseen exercise]**

(e) Theorem 2 and the result from (d) tell us that the function g is differentiable on $(-1, 1)$ and that its derivative is obtained by differentiating the power series for g term-by-term:

$$g'(x) = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \right)' = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1} x^{2k+1} \right)' = \sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2}$$

because we are looking at a geometric series. **[4 marks; unseen exercise]**

(f) Application of the result from (a) with $f(x) = \arctan x$ and $g(x)$ as in (e) gives

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + c, \qquad \forall x \in (-1, 1).$$

Setting $x = 0$ we conclude that $c = 0$. **[2 marks; unseen exercise]**

Solution to question 5

(a)

Theorem 3 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}. \tag{11}$$

[4 marks; bookwork]

(b) Fix $x \in (a, 0)$ and consider the function

$$g(t) := f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - \dots$$
$$- \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x-t)^n \,, \quad (12)$$

where t is the independent variable. Since $f : (a, b) \to \mathbb{R}$ is $n+1$ times differentiable, $g : (a, b) \to \mathbb{R}$ is differentiable. Differentiating (12) we get

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \dots$$
$$\dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right]$$
$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

(all the terms cancelled out, apart from the last one). Applying Theorem 3 to the function $g(t)$ on the interval $[x, 0]$ (which we can because g is differentiable on (a, b), hence continuous on $[x, 0] \subset (a, b)$ and differentiable on $(x, 0) \subset (a, b)$) we get

$$\frac{g(0) - g(x)}{0 - x} = g'(\xi) = -\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$$

for some $\xi \in (x, 0)$. But

$$g(0) = R_n(x), \quad g(x) = 0,$$

so the above formula can be rewritten as

$$\frac{R_n(x)}{-x} = -\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n \,,$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n x$. $\square$ **[12 marks; bookwork; note that in the lectures the above argument was done briefly (I did in detail Lagrange's form of the remainder) and only for the case $x > 0$]**

(c) Application of the result from (b) to the function $f : (-1, 99) \to \mathbb{R}$, $f(x) = \ln(1 + x)$ with $x = -\frac{2}{3}$ gives

$$\ln\left(\frac{1}{3}\right) = -\sum_{k=1}^{n} \frac{2^n}{n3^n} + R_n\left(-\frac{2}{3}\right),$$

$$R_n\left(-\frac{2}{3}\right) = (-1)^n \frac{\left(-\frac{2}{3} - \xi\right)^n \left(-\frac{2}{3}\right)}{(1+\xi)^{n+1}} = -\frac{\frac{2}{3}}{1+\xi}\left(\frac{\frac{2}{3}+\xi}{1+\xi}\right)^n$$

for some $\xi \in (-2/3, 0)$. We have

$$\max_{\xi \in [-\frac{2}{3}, 0]} \frac{\frac{2}{3}+\xi}{1+\xi} = \frac{2}{3}$$

(here the function being maximised has no critical points and achieves its maximum at the endpoint $\xi = 0$), so

$$\left|R_n\left(-\frac{2}{3}\right)\right| \le \frac{\left(\frac{2}{3}\right)^{n+1}}{1+\xi} \le 3\left(\frac{2}{3}\right)^{n+1} \to 0 \qquad \text{as} \qquad n \to \infty,$$

and, hence,

$$R_n\left(-\frac{2}{3}\right) \to 0 \qquad \text{as} \qquad n \to \infty,$$

which gives the required result. **[9 marks; this question is a special case of a question from an exercise sheet]**