All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. In this question $\{x_n\}$ is a sequence of real numbers.

   (a) Give the mathematical definition of the statement "The sequence $\{x_n\}$ converges to a limit L".

   (b) Give the mathematical definition of the statement "$\{x_n\}$ is a Cauchy sequence".

   (c) State Cauchy's General Principle of Convergence for $\{x_n\}$.

   (d) Suppose that $|x_{n+1} - x_n| \leq \dfrac{1}{n(n+1)}$ for all $n \in \mathbb{N}$.

   (i) Prove that $|x_m - x_n| < \dfrac{1}{n}$ for $m > n$.

   (ii) Prove that $|x_m - x_n| < \dfrac{1}{m} + \dfrac{1}{n}$ for all $m, n \in \mathbb{N}$.

   (iii) Prove that the sequence $\{x_n\}$ converges.

2. (a) What is a partition P of an interval $[a, b]$ and what is its mesh $\|P\|$?

   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded.

   (i) Define $L(f, P)$, the lower Darboux sum of f with respect to the given partition P of the interval $[a, b]$.

   (ii) Define $\underline{\int}_a^b f(x)\, dx$, the lower Riemann integral of f over $[a, b]$.

   (c) Consider the function $f : [0, 1] \to \mathbb{R}$, $f(x) = x$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[0, 1]$.

   (i) Prove that for all $i = 1, 2, \ldots, n$ we have

   $$\frac{1}{2}(x_i^2 - x_{i-1}^2) - \frac{1}{2}(x_i - x_{i-1})\|P\| \leq \left(\inf_{x\in[x_{i-1}, x_i]} f(x)\right)(x_i - x_{i-1}) < \frac{1}{2}(x_i^2 - x_{i-1}^2)\,.$$

   (ii) Prove that $\frac{1}{2}(1 - \|P\|) \leq L(f, P) < \frac{1}{2}$.

   (d) Use the definition from (b)(ii) and the result from (c)(ii) to prove that $\underline{\int}_0^1 x\, dx = \frac{1}{2}$.

3. (a) Let $f : [a, b] \to \mathbb{R}$ be bounded. Suppose that for any $\varepsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Prove that f is Riemann integrable on $[a, b]$.

Here $L(f, P)$ and $U(f, P)$ are the lower and upper Darboux sums of f with respect to the partition P. In doing this part of the question you may use without proof the fact that the lower Riemann integral is less than or equal to the upper Riemann integral.

(b) Give the mathematical definition of the statement "The function $f : [a, b] \to \mathbb{R}$ is uniformly continuous".

(c) Let $f : [a, b] \to \mathbb{R}$ be continuous. Using the result from (a), prove that f is Riemann integrable on $[a, b]$.

In doing this part of the question you may use without proof the fact that if $f : [a, b] \to \mathbb{R}$ is continuous then it is uniformly continuous.

(d) Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which has an infinite number of discontinuities but is Riemann integrable on $[0, 1]$. Briefly justify your answer. You may use any result from the course.

4. (a) Suppose that the functions $f, g : (-1, 1) \to \mathbb{R}$ are differentiable and that $f'(x) = g'(x)$ for all $x \in (-1, 1)$. Prove that $f(x) = g(x) + c$ for all $x \in (-1, 1)$, where c is a constant.

[*Hint:* you may find it helpful to use the Mean Value Theorem.]

(b) Define the notion of the radius of convergence of a real power series.

(c) State the theorem about the differentiability of a real power series.

(d) Find the radius of convergence of the real power series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$.

(e) Consider the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. Prove that

$$g'(x) = \frac{1}{1+x^2}.$$

(f) Prove that $\arctan x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ for all $x \in (-1, 1)$. Here you may use without proof the fact that $\arctan' x = \dfrac{1}{1+x^2}$.

[*Hint:* you may find it helpful to use the results from (a) and (e).]

5. (a) State the Mean Value Theorem.

(b) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a, b) \to \mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

Given an $x \in (a, 0)$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n x$ for some $\xi \in (x, 0)$.

(c) Use the result from (b) to prove that $\ln\left(\dfrac{1}{3}\right) = -\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n\,3^n}$.

$\Big[$ *Hint:* you may find it helpful to evaluate $\displaystyle\max_{\xi \in [-\frac{2}{3}, 0]} \frac{\frac{2}{3} + \xi}{1 + \xi}$. $\Big]$