MATH1102 Analysis

Solutions to examination paper for the academic year 2017/2018

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Solution to question 1

(a)

Definition 1 *We say that the function* $f : (0, +\infty) \to \mathbb{R}$ *is* continuous at the point $x_0 \in (0, +\infty)$ *if* $\forall \varepsilon > 0 \quad \exists \delta > 0$ *such that*

$$|x - x_0| < \delta \quad \& \quad x \in (0, +\infty) \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

[4 marks; bookwork]

(b)

Definition 2 *We say that the function* $f : (0, +\infty) \to \mathbb{R}$ *is* continuous *if it is continuous at every point* $x_0 \in (0, +\infty)$.

[2 marks; bookwork]

(c)

Definition 3 *We say that the function* $f : (0, +\infty) \to \mathbb{R}$ *is* uniformly continuous *if* $\forall \varepsilon > 0 \quad \exists \delta > 0$ *such that*

$$|x - y| < \delta \quad \& \quad x, y \in (0, +\infty) \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon.$$

[4 marks; bookwork]

(d) Let ε be an arbitrary positive number. Take arbitrary x, y such that

$$|x - y| < \varepsilon \quad \& \quad x, y \in (0, +\infty). \tag{1}$$

I claim that (1) implies

$$\left| \frac{1}{1+x} - \frac{1}{1+y} \right| < \varepsilon. \tag{2}$$

Indeed, if $x = y$ then (2) is, obviously, true. If $x \neq y$ then the Mean Value Theorem tells us that

$$\left| \frac{1}{1+x} - \frac{1}{1+y} \right| = \frac{|x-y|}{(1+\xi)^2} \tag{3}$$

for some ξ strictly between x and y. But $\xi > 0$, so $\frac{1}{(1+\xi)^2} < 1$ and, consequently, (3) and (1) imply (2). Thus, the definition of uniform continuity works with $\delta = \varepsilon$. **[6 marks; unseen exercise]**

(e) Suppose that our function $f : (0, +\infty) \to (0, +\infty)$, $f(x) = \dfrac{1}{x}$, is uniformly continuous. Applying the definition of uniform continuity with $\varepsilon = 1$ we get a $\delta > 0$ such that

$$|x - y| < \delta \quad \& \quad x, y \in (0, +\infty) \qquad \Longrightarrow \qquad \left| \frac{1}{x} - \frac{1}{y} \right| < 1. \tag{4}$$

Put $x = \frac{1}{n+1}$ and $y = \frac{1}{n}$, where $n \in \mathbb{N}$. We have

$$|x - y| = y - x = \frac{1}{n(n+1)} < \frac{1}{n},$$

so if we choose $n > \frac{1}{\delta}$ we are guaranteed to have $|x - y| < \delta$. But

$$\left| \frac{1}{x} - \frac{1}{y} \right| = |(n+1) - n| = 1,$$

contradicting (4). **[9 marks; unseen exercise]**

Solution to question 2

(a)

Definition 4

(i) A partition *P of the interval* $[a, b]$ *is a finite sequence of real numbers* $P = \{x_0, x_1, \ldots, x_n\}$ *such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The set of all partitions of a given interval $[a, b]$ *will be denoted by* $\mathcal{P}[a, b]$.

(ii) The mesh (norm, width) *of the partition* $P = \{x_0, x_1, \ldots, x_n\}$ *is the number*

$$\|P\| := \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

[3 marks; bookwork]

(b)(i) The *lower Darboux sum* of f with respect to the partition $P = \{x_0, x_1, \ldots, x_n\}$ is defined as

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

where $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$. The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i := \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$. **[3 marks; bookwork]**

(b)(ii)

Definition 5 *Let* $f \in \mathcal{B}[a, b]$. *The* lower Riemann integral *of* f *over* $[a, b]$ *is defined as*

$$\underline{\int_a^b} f(x)\, dx := \sup_{P \in \mathcal{P}[a,b]} L(f, P).$$

The upper Riemann integral *of* f *over* $[a, b]$ *is defined as*

$$\overline{\int_a^b} f(x)\, dx := \inf_{P \in \mathcal{P}[a,b]} U(f, P).$$

[3 marks; bookwork]

(b)(iii) For any pair of partitions P and Q we have $L(f,P) \le U(f,Q)$. Taking the supremum over $P \in \mathcal{P}[a,b]$ we get

$$\underline{\int_a^b} f(x)\,dx \le U(f,Q). \tag{5}$$

Note that formula (5) holds for any partition Q. Taking the infimum over $Q \in \mathcal{P}[a,b]$ we get $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$. **[3 marks; bookwork]**

(b)(iv)

Definition 6 *The function $f \in \mathcal{B}[a,b]$ is said to be* Riemann integrable *on $[a,b]$ if*

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx\,.$$

In this case the common value of the lower and upper Riemann integrals is denoted by

$$\int_a^b f(x)\,dx$$

and is called the Riemann integral *of f over $[a,b]$. The set of all Riemann integrable functions $f:[a,b]\to\mathbb{R}$ will be denoted by $\mathcal{R}[a,b]$.*

[2 marks; bookwork]

(c)(i) We have

$$L(f,P_n) = \frac{1}{n^2}\sum_{i=1}^{n}(i-1) = \frac{1}{n^2}\left(\sum_{i=1}^{n} i - n\right) = \frac{1}{n^2}\left(\frac{n(n+1)}{2} - n\right) = \frac{1}{2} - \frac{1}{2n}\,,$$

$$U(f,P_n) = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n}\,.$$

[5 marks; unseen exercise]

(c)(ii) In what follows the P_n are the particular partitions from (c)(i). Using the definitions from (b)(ii) and the results from (c)(i) we get

$$\underline{\int_0^1} x\,dx := \sup_{P\in\mathcal{P}[0,1]} L(f,P) \ge \sup_{n\in\mathbb{N}} L(f,P_n) = \sup_{n\in\mathbb{N}}\left(\frac{1}{2} - \frac{1}{2n}\right) = \frac{1}{2}\,,$$

$$\overline{\int_0^1} x\,dx := \inf_{P\in\mathcal{P}[0,1]} U(f,P) \le \inf_{n\in\mathbb{N}} U(f,P_n) = \inf_{n\in\mathbb{N}}\left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2}\,.$$

[4 marks; unseen exercise]

(c)(iii) Combining the results from (b)(iii) and (c)(ii) we get

$$\frac{1}{2} \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le \frac{1}{2}$$

which implies

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \frac{1}{2}\,.$$

According to the definition from (b)(iv) this means that $\int_0^1 x\,dx = \frac{1}{2}$. **[2 marks; unseen exercise]**

Solution to question 3

(a)

Definition 7 *Let $f : [a, b] \to \mathbb{R}$. A* primitive *of f is a function $F : [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$, differentiable on (a, b) and satisfies $F' = f$ on (a, b).*

[3 marks; bookwork]

(b) No: one can always add a constant to F. **[1 mark; bookwork]**

(c)

Theorem 1 *Let $f \in \mathcal{R}[a, b]$ and let F be a primitive of f. Then*

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a) \;=\; F(x)\big|_a^b\,. \tag{6}$$

[4 marks; bookwork]

Note: $F(x)\big|_a^b$ is simply a short way of writing $F(b) - F(a)$.

Proof of Theorem 1 Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Applying the Mean Value Theorem to the function F on the subinterval $[x_{i-1}, x_i]$ we get

$$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1})$$

for some $t_i \in (x_{i-1}, x_i)$. But

$$m_i \le f(t_i) \le M_i$$

where I use the standard notation $m_i := \inf\limits_{x\in[x_{i-1},x_i]} f(x)$, $M_i := \sup\limits_{x\in[x_{i-1},x_i]} f(x)$. Hence,

$$m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1}).$$

Summing up over i from 1 to n we get

$$L(f, P) \le \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \le U(f, P).$$

But

$$\sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(x_n) - F(x_0) = F(b) - F(a),$$

so the above inequality can be rewritten as

$$L(f, P) \le F(b) - F(a) \le U(f, P). \tag{7}$$

The inequality (7) is true for any partition P. Taking a limiting sequence of partitions $\{P_k\}$ and letting $k \to \infty$ we get from (7)

$$\lim_{k\to\infty} L(f, P_k) \le F(b) - F(a) \le \lim_{k\to\infty} U(f, P_k). \tag{8}$$

It remains to note that as $f \in \mathcal{R}[a, b]$ the result stated in the examination paper tells us that

$$\lim_{k\to\infty} L(f, P_k) = \lim_{k\to\infty} U(f, P_k) = \int_a^b f(x)\,dx\,. \tag{9}$$

Substituting (9) into (8), we get

$$\int_a^b f(x)\,dx \le F(b) - F(a) \le \int_a^b f(x)\,dx$$

which implies (6). $\square$ **[8 marks; bookwork]**

Remark 1 *The Fundamental Theorem of Calculus can be proven without the use of a limiting sequence of partitions. It suffices to use the definition of the Riemann integral. Such a solution would carry full marks.*

(d)(i) We have

$$-x^2 \le F(x) \le x^2, \qquad \forall x \in (0,1],$$
$$\lim_{x\to 0^+} x^2 = \lim_{x\to 0^+} (-x^2) = 0 = F(0),$$

so the Sandwich Principle tells us that

$$\lim_{x\to 0^+} F(x) = F(0),$$

which means that the function F is continuous at the point $x_0 = 0$. **[5 marks; unseen exercise]**

(d)(ii) The function F is continuous on $(0,1]$ as a composition of continuous functions. It is also continuous at the point $x_0 = 0$ by (d)(i). Hence, the function F is continuous on $[0,1]$.

The function F is differentiable on $(0,1)$ as a composition of differentiable functions. Elementary calculations show that we have $F'(x) = f(x)$ on $(0,1)$.

Thus, the function F satisfies the definition of a primitive from (a). **[3 marks; unseen exercise]**

(d)(iii) The Fundamental Theorem of Calculus gives us

$$\int_0^1 f(x)\,dx = F(1) - F(0) = \cos 1.$$

[1 marks; unseen exercise]

Solution to question 4

(a)

Theorem 2 *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $\xi \in (a,b)$ such that*

$$(g(b) - g(a))f'(\xi) = (f(b) - f(a))g'(\xi)\,. \tag{10}$$

[3 marks; bookwork]

(b)

Theorem 3 *Let $f, g : (a,b) \to \mathbb{R}$ be differentiable and let $c \in (a,b)$ be such that $f(c) = g(c) = 0$ and $g'(x) \ne 0$ for $x \ne c$. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{11}$$

provided the latter limit exists.

[4 marks; bookwork]

Explanation why $g(x) \ne 0$ for $x \ne c$ (this is not part of the proof of the theorem). Indeed, suppose $g(x) = 0$ for some $x \in (a,b)$, $x \ne c$.

Case 1: $x > c$. Apply Rolle's Theorem to the function g on the interval $[c,x]$: Rolle's Theorem tells us that there exists a $\xi \in (c,x)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 3.

Case 2: $x < c$. Apply Rolle's Theorem to the function g on the interval $[x, c]$: Rolle's Theorem tells us that there exists a $\xi \in (x, c)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 3.

Proof of Theorem 3 Suppose $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists and equals L. Then

$$\lim_{x\to c^+} \frac{f'(x)}{g'(x)} = L \tag{12}$$

and

$$\lim_{x\to c^-} \frac{f'(x)}{g'(x)} = L. \tag{13}$$

In order to prove (11) we need to show that

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = L \tag{14}$$

and

$$\lim_{x\to c^-} \frac{f(x)}{g(x)} = L. \tag{15}$$

Let us prove (14).

Take an arbitrary sequence $\{x_n\} \subset (c, b)$ such that

$$\lim_{n\to\infty} x_n = c. \tag{16}$$

Applying Theorem 2 on the interval $[c, x_n]$ we get a sequence $\{\xi_n\}$ such that

$$c < \xi_n < x_n \tag{17}$$

and

$$(g(x_n) - g(c))f'(\xi_n) = (f(x_n) - f(c))g'(\xi_n)$$

which can be equivalently rewritten as

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(\xi_n)}{g'(\xi_n)}. \tag{18}$$

Formulae (16), (17) and the Sandwich Principle imply

$$\lim_{n\to\infty} \xi_n = c. \tag{19}$$

Formulae (12), (19) and the sequential definition of a limit (one-sided version) imply

$$\lim_{n\to\infty} \frac{f'(\xi_n)}{g'(\xi_n)} = L. \tag{20}$$

Formulae (18) and (20) imply

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L. \tag{21}$$

Formula (21) and the sequential definition of a limit (one-sided version) imply[1] (14).

Formula (15) is proved similarly. $\square$ **[9 marks; bookwork]**

(c)(i) Applying L'Hôpital's Rule once we get

$$\lim_{x\to 0} \frac{\sinh x}{x} = \lim_{x\to 0} \cosh x = \cosh 0 = 1.$$

[1]Here it is important that the sequence $\{x_n\} \subset (c, b)$ is an *arbitrary* sequence satisfying (16).

[2 marks; unseen exercise]

(c)(ii) Applying L'Hôpital's Rule once and using the result from (c)(i) we get

$$\lim_{x\to 0} \frac{1-\cosh x}{x^2} = -\frac{1}{2} \lim_{x\to 0} \frac{\sinh x}{x} = -\frac{1}{2}\,.$$

[2 marks; unseen exercise]

(c)(iii) Using the result from (c)(i) we get

$$\begin{aligned} \lim_{x\to 0} \frac{\sinh^2 x - \tanh^2 x}{x^4} &= \lim_{x\to 0} \frac{\sinh^2(\cosh^2 x - 1)}{x^4 \cosh^2 x} = \lim_{x\to 0} \frac{\sinh^4 x}{x^4 \cosh^2 x} \\ &= \left(\lim_{x\to 0} \frac{1}{\cosh x}\right)^2 \left(\lim_{x\to 0} \frac{\sinh x}{x}\right)^4 = \left(\frac{1}{\cosh 0}\right)^2 (1)^4 = 1. \end{aligned}$$

[5 marks; unseen exercise]

Solution to question 5

(a)

Definition 8 *A function $f : [1, +\infty) \to \mathbb{R}$ is said to be* locally Riemann integrable *over* $[1, +\infty)$ *if $\forall a, b \in [1, +\infty)$ we have $f \in \mathcal{R}[a, b]$. The set of all locally Riemann integrable functions over the interval* $[1, +\infty)$ *will be denoted by* $\mathcal{R}_{loc}[1, +\infty)$.

[2 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval I and not specifically for the interval $[1, +\infty)$]

(b) We say that f is integrable on $[1, +\infty)$ in the improper sense if $\lim\limits_{b\to+\infty} \int_1^b f(x)\,dx$ exists. We will denote this limit by $\displaystyle\int_1^{+\infty} f(x)\,dx$. **[4 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval I and not specifically for the interval $[1, +\infty)$]**

(c) Note that the integrand is continuous on $[1, +\infty)$, hence locally integrable on $[1, +\infty)$. So we have to check only integrability at $+\infty$. We have

$$\int_1^b \frac{1}{x^{3/2}}\,dx = 2 - \frac{2}{b^{1/2}} \to 2 \qquad \text{as} \qquad b \to +\infty.$$

Hence,

$$\int_1^{+\infty} \frac{1}{x^{3/2}}\,dx = 2.$$

[2 marks; special case of an argument done in the lectures]

(d)

Theorem 4 *Let $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d \rangle$, and suppose that*

- *$0 \le f(x) \le g(x)$ for all $x \in I$;*
- *$\int_c^d g(x)\,dx$ exists.*

Then $\int_c^d f(x)\,dx$ exists.

[3 marks; bookwork]

(e)

Theorem 5 *Suppose that $f \in \mathcal{R}_{loc}(I)$ and $|f|$ is integrable over I. Then f is integrable over I.*

[**3 marks; bookwork**]

(f) Let us prove the existence of the improper integral

$$\int_1^{+\infty} \frac{\sin x}{x^{3/2}}\, dx\,. \tag{22}$$

Note that the integrand is continuous on $[1, +\infty)$, hence locally integrable on $[1, +\infty)$. So we have to check only integrability at $+\infty$.

By Theorem 5, in order to prove the existence of the integral (22) it is sufficient to prove the existence of the integral

$$\int_1^{+\infty} \frac{|\sin x|}{x^{3/2}}\, dx\,. \tag{23}$$

Compare now the functions $\dfrac{|\sin x|}{x^{3/2}}$ and $\dfrac{1}{x^{3/2}}$ on the interval $[1, +\infty)$. We have

$$0 \le \frac{|\sin x|}{x^{3/2}} \le \frac{1}{x^{3/2}}\,,$$

so, by Theorem 4, in order to prove the existence of the integral (23) it is sufficient to prove the existence of the integral $\int_1^{+\infty} \frac{1}{x^{3/2}}\, dx$. But this has already been done in (c). [**6 marks; unseen exercise**]

(g) Note that the integrand is continuous on $[1, +\infty)$, hence locally integrable on $[1, +\infty)$. So we have to check only integrability at $+\infty$.

Integrating by parts we get

$$\int_1^b \frac{\cos x}{x^{1/2}}\, dx = \frac{\sin b}{b^{1/2}} - \sin 1 + \frac{1}{2} \int_1^b \frac{\sin x}{x^{3/2}}\, dx\,.$$

We have $\frac{\sin b}{b^{1/2}} - \sin 1 \to -\sin 1$ as $b \to +\infty$, so in order to prove the existence of the original integral we have to prove the existence of the integral $\int_1^{+\infty} \frac{\sin x}{x^{3/2}}\, dx$. But this has already been done in (f).

[**5 marks; unseen exercise**]