All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Give, in terms of ε and δ, the mathematical definition of the statement "the function $f : (0, +\infty) \to \mathbb{R}$ is continuous at the point $x_0 \in (0, +\infty)$".
   (b) Give the mathematical definition of the statement "the function $f : (0, +\infty) \to \mathbb{R}$ is continuous".
   (c) Give, in terms of ε and δ, the mathematical definition of the statement "the function $f : (0, +\infty) \to \mathbb{R}$ is uniformly continuous".
   (d) Prove that the function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = \frac{1}{1+x}$, is uniformly continuous.
   (e) Prove that the function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is not uniformly continuous.

2. (a) What is a partition of an interval $[a, b]$ and what is its mesh?
   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded.
      (i) Given a partition P of the interval $[a, b]$, define the lower Darboux sum $L(f, P)$ and the upper Darboux sum $U(f, P)$.
      (ii) Define the lower Riemann integral $\underline{\int}_a^b f(x)\,dx$ and the upper Riemann integral $\overline{\int}_a^b f(x)\,dx$.
      (iii) Prove that $\underline{\int}_a^b f(x)\,dx \le \overline{\int}_a^b f(x)\,dx$. Here you may use without proof the fact that $L(f, P) \le U(f, Q)$ for any pair of partitions P and Q.
      (iv) Give the mathematical definition of the statement "the function f is Riemann integrable on $[a, b]$". Define the Riemann integral $\int_a^b f(x)\,dx$.
   (c) Consider the function $f : [0, 1] \to \mathbb{R}$, $f(x) = x$, and let $P_n = \{x_0, x_1, \ldots, x_n\}$ be the partition of the interval $[0, 1]$ into n subintervals of equal length, that is, $x_i = \frac{i}{n}$, $i = 0, 1, \ldots, n$.
      (i) Evaluate the Darboux sums $L(f, P_n)$ and $U(f, P_n)$.
      (ii) Use the definitions from (b)(ii) and the results from (c)(i) to prove that $\underline{\int}_0^1 x\,dx \ge \frac{1}{2}$ and $\overline{\int}_0^1 x\,dx \le \frac{1}{2}$.
      (iii) Use the definition from (b)(iv) and the results from (b)(iii) and (c)(ii) to prove that $\int_0^1 x\,dx = \frac{1}{2}$.

3. (a) Give the mathematical definition of the statement "the function $F : [a, b] \to \mathbb{R}$ is a primitive of the function $f : [a, b] \to \mathbb{R}$".

(b) Suppose that the function $f : [a, b] \to \mathbb{R}$ has a primitive $F : [a, b] \to \mathbb{R}$. Is this primitive unique?

(c) State and prove the Fundamental Theorem of Calculus.

Here you may use without proof the following fact: if the function f is Riemann integrable on the interval $[a, b]$ and $\{P_k\}$ is a sequence of partitions such that $\lim_{k\to\infty} \|P_k\| = 0$, then $\lim_{k\to\infty} L(f, P_k) = \lim_{k\to\infty} U(f, P_k) = \int_a^b f(x)\,dx$.

(d) Consider the functions $f, F : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin\frac{1}{x} + 2x\cos\frac{1}{x} & \text{if} \quad x \neq 0, \\ 5 & \text{if} \quad x = 0, \end{cases} \qquad F(x) = \begin{cases} x^2\cos\frac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

(i) Prove that the function F is continuous at the point $x_0 = 0$.

(ii) Prove that the function F is a primitive of the function f.

(iii) Evaluate $\int_0^1 f(x)\,dx$. Here you may use without proof the fact that the function f is Riemann integrable on the interval $[0, 1]$.

4. (a) State Cauchy's Generalisation of the Mean Value Theorem.

(b) State and prove l'Hôpital's Rule for finding $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ when $f(c) = g(c) = 0$.

(c) Evaluate the following limits:

(i) $\lim\limits_{x\to 0} \dfrac{\sinh x}{x}$,

(ii) $\lim\limits_{x\to 0} \dfrac{1 - \cosh x}{x^2}$,

(iii) $\lim\limits_{x\to 0} \dfrac{\sinh^2 x - \tanh^2 x}{x^4}$.

5. (a) Give the mathematical definition of the statement "the function $f : [1, +\infty) \to \mathbb{R}$ is locally Riemann integrable".

(b) Let $f : [1, +\infty) \to \mathbb{R}$ be locally Riemann integrable. Give the mathematical definition of the statement "the function f is integrable over $[1, +\infty)$ in the improper sense". Define the improper integral $\displaystyle\int_1^{+\infty} f(x)\, dx$.

(c) Use the definition from (b) to prove the existence of the improper integral $\displaystyle\int_1^{+\infty} \frac{1}{x^{3/2}}\, dx$ and to evaluate this integral.

(d) State the Comparison Theorem for Improper Integrals.

(e) State the theorem relating the integrability, in the improper sense, of the functions f and $|f|$.

(f) Prove the existence of the improper integral $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^{3/2}}\, dx$.

[*Hint:* you may find it helpful to use the results from (c), (d) and (e).]

(g) Prove the existence of the improper integral $\displaystyle\int_1^{+\infty} \frac{\cos x}{x^{1/2}}\, dx$.

[*Hint:* you may find it helpful to integrate by parts.]