MATH1102 Analysis

Solutions to examination paper for the academic year 2016/2017

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Solution to question 1

(a)

Definition 1 *We say that the function* $f : [0, +\infty) \to \mathbb{R}$ *is* continuous at the point $x_0 \in [0, +\infty)$ *if* $\forall \varepsilon > 0 \;\; \exists \delta > 0$ *such that*

$$|x - x_0| < \delta \quad \& \quad x \in [0, +\infty) \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

[4 marks; bookwork]

(b)

Definition 2 *We say that the function* $f : [0, +\infty) \to \mathbb{R}$ *is* continuous *if it is continuous at every point* $x_0 \in [0, +\infty)$.

[2 marks; bookwork]

(c)

Definition 3 *We say that the function* $f : [0, +\infty) \to \mathbb{R}$ *is* uniformly continuous *if* $\forall \varepsilon > 0 \;\; \exists \delta > 0$ *such that*

$$|x - y| < \delta \quad \& \quad x, y \in [0, +\infty) \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon.$$

[4 marks; bookwork]

(d) Let ε be an arbitrary positive number. Take arbitrary x, y such that

$$|x - y| < \varepsilon \quad \& \quad x, y \in [0, +\infty). \tag{1}$$

I claim that (1) implies

$$|e^{-x} - e^{-y}| < \varepsilon. \tag{2}$$

Indeed, if $x = y$ then (2) is, obviously, true. If $x \neq y$ then the Mean Value Theorem tells us that

$$|e^{-x} - e^{-y}| = e^{-\xi}|x - y| \tag{3}$$

for some ξ strictly between x and y. But $\xi > 0$, so $e^{-\xi} < 1$ and, consequently, (3) and (1) imply (2). Thus, the definition of uniform continuity works with $\delta = \varepsilon$. **[6 marks; unseen exercise]**

(e) Suppose our function is uniformly continuous. Applying the definition of uniform continuity with $\varepsilon = 1$ we get a $\delta > 0$ such that

$$|x-y| < \delta \quad \& \quad x, y \in [0, +\infty) \qquad \Longrightarrow \qquad |e^x - e^y| < 1. \tag{4}$$

Choose a $z \in [0, +\infty)$ such that

$$e^z > \frac{2}{\delta} \tag{5}$$

and put $x = z$, $y = z + \frac{\delta}{2}$. The Mean Value Theorem tells us that

$$|e^x - e^y| = e^\xi |x-y| = \frac{\delta}{2} e^\xi \tag{6}$$

for some $\xi \in (z, z+\frac{\delta}{2})$. We have $\xi > z$, so $e^\xi > e^z$ and, consequently, (6) and (5) imply $|e^x - e^y| > 1$, contradicting (4). **[9 marks; unseen exercise]**

Solution to question 2

(a)

Definition 4

(i) A partition P *of the interval* $[a,b]$ *is a finite sequence of real numbers* $P = \{x_0, x_1, \ldots, x_n\}$ *such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The set of all partitions of a given interval $[a,b]$ *will be denoted by* $\mathcal{P}[a,b]$.

(ii) The mesh (norm, width) *of the partition* $P = \{x_0, x_1, \ldots, x_n\}$ *is the number*

$$\|P\| := \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

[3 marks; bookwork]

(b)(i) The *lower Darboux sum* of f with respect to the partition $P = \{x_0, x_1, \ldots, x_n\}$ is defined as

$$L(f,P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

where $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$. The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f,P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i := \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$. **[3 marks; bookwork]**

(b)(ii)

Definition 5 *Let* $f \in \mathcal{B}[a,b]$. *The* lower Riemann integral *of* f *over* $[a,b]$ *is defined as*

$$\underline{\int_a^b} f(x)\, dx := \sup_{P \in \mathcal{P}[a,b]} L(f,P).$$

The upper Riemann integral *of* f *over* $[a,b]$ *is defined as*

$$\overline{\int_a^b} f(x)\, dx := \inf_{P \in \mathcal{P}[a,b]} U(f,P).$$

[3 marks; bookwork]

(b)(iii)

Definition 6 *The function* $f \in \mathcal{B}[a,b]$ *is said to be* Riemann integrable *on* $[a,b]$ *if*

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx\,.$$

In this case the common value of the lower and upper Riemann integrals is denoted by

$$\int_a^b f(x)\,dx$$

and is called the Riemann integral *of* f *over* $[a,b]$*. The set of all Riemann integrable functions* $f : [a,b] \to \mathbb{R}$ *will be denoted by* $\mathcal{R}[a,b]$*.*

[1 mark; bookwork]

(b)(iv)

Theorem 1 *Let* $f \in \mathcal{B}[a,b]$*. Then* $f \in \mathcal{R}[a,b]$ *iff* $\forall \varepsilon > 0 \;\; \exists P \in \mathcal{P}[a,b]$ *such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{7}$$

[3 marks; bookwork]

(c)

Definition 7 *A function* $f : [a,b] \to \mathbb{R}$ *is said to be* decreasing *on* $[a,b]$ *if*

$$x_1 < x_2 \quad \& \quad x_1, x_2 \in [a,b] \qquad \Longrightarrow \qquad f(x_1) \geq f(x_2).$$

[1 mark; bookwork]

(d) We have $f(a) \geq f(x) \geq f(b)$, so our decreasing function is bounded. **[1 mark; for the case of an increasing function this argument was done in the lectures]**

(e) Consider a partition P_n into n subintervals of equal length: $P_n = \{x_0, x_1, \ldots, x_n\}$,

$$x_i = a + \frac{b-a}{n} i, \qquad i = 0, 1, \ldots, n.$$

Then

$$L(f, P_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i), \qquad U(f, P_n) = \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i),$$

$$U(f,P_n) - L(f,P_n) = \frac{b-a}{n}(f(x_0) - f(x_n)) = \frac{b-a}{n}(f(a) - f(b))\,.$$

Now, let ε be an arbitrary positive number. Choose n so large that

$$\frac{b-a}{n}(f(a) - f(b)) < \varepsilon.$$

Then

$$U(f,P_n) - L(f,P_n) < \varepsilon$$

and Riemann's Criterion for Integrability tells us that f is Riemann integrable on $[a,b]$. **[7 marks; for the case of an increasing function this proof was done in the lectures]**

(f) The function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x > 0 \end{cases}$$

is not continuous. However, it is decreasing, so by the result from (e) it is integrable on $[0,1]$. **[3 marks; unseen exercise]**

Solution to question 3

(a)(i) The integral makes sense because the function is monotonic. **[2 marks; bookwork]**

(a)(ii) Put

$$T_n := \sum_{k=r}^{n} f(k) - \int_r^n f(x)\,dx \tag{8}$$

for $n \in \mathbb{N}$, $n \geq r$.

We have

$$\begin{aligned} T_n - T_{n+1} = \int_n^{n+1} f(x)\,dx - f(n+1) = \int_n^{n+1} f(x)\,dx - \int_n^{n+1} f(n+1)\,dx \\ = \int_n^{n+1} (f(x) - f(n+1))\,dx \geq 0, \end{aligned}$$

so the sequence $\{T_n\}$ is decreasing. **[4 marks; bookwork]**

(a)(iii) Now I estimate T_n from below:

$$T_n = \sum_{k=r}^{n-1} \int_k^{k+1} f(k)\,dx + f(n) - \sum_{k=r}^{n-1} \int_k^{k+1} f(x)\,dx = \sum_{k=r}^{n-1} \int_k^{k+1} (f(k) - f(x))\,dx + f(n) \geq f(n).$$

[8 marks; bookwork]

(a)(iv) The above estimate and the fact that f is nonnegative imply

$$T_n \geq 0.$$

[1 mark; bookwork]

(a)(v) We are looking at a decreasing sequence $\{T_n\}$ which is bounded from below by zero. Such a sequence has to converge. **[2 marks; bookwork]**

(a)(vi) Suppose that the series $\sum\limits_{k=r}^{\infty} f(k)$ converges. Formula (8) can be rewritten as

$$\int_r^n f(x)\,dx = \sum_{k=r}^{n} f(k) - T_n\,.$$

The fact that the sequence $\{T_n\}$ converges and the algebra of limits tell us that $\lim_{n\to\infty} \int\limits_r^n f(x)\,dx$ exists.

Conversely, suppose that $\lim_{n\to\infty}\int\limits_r^n f(x)\,dx$ exists. Formula (8) can be rewritten as

$$\sum_{k=r}^{n} f(k) = \int_r^n f(x)\,dx + T_n\,.$$

The fact that the sequence $\{T_n\}$ converges and the algebra of limits tell us that the series $\sum\limits_{k=r}^{\infty} f(k)$ converges. **[3 marks; bookwork]**

(b) Let us introduce the function $f:[2,+\infty)\to[0,+\infty)$, $f(x):=\dfrac{1}{x\ln x}$. This function is decreasing and takes nonnegative values, so by the Integral Test for the Convergence of Series (result from (a)(vi)) our series converges if and only if

$$\lim_{n\to\infty}\int_2^n \frac{dx}{x\ln x}$$

exists. But

$$\int_2^n \frac{dx}{x\ln x} = \ln(\ln x)|_{x=2}^{x=n} = \ln(\ln n)-\ln(\ln 2)\to+\infty \qquad \text{as} \qquad n\to\infty.$$

Hence, the series diverges. **[5 marks; unseen exercise]**

Solution to question 4

(a) Let us introduce the set

$$S:=\left\{r\geq 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ converges for } x=r \text{ or } x=-r\right\}. \tag{9}$$

Obviously, $\{0\}\subset S\subset[0,+\infty)$.

Definition 8 *The* radius of convergence *of a power series is the extended real number*

$$R:=\sup S$$

where S is the set (9).

[3 marks; bookwork]

(b) The two power series,

$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n x^{n-k}\,.$$

have the same radii of convergence. **[2 marks; bookwork]**

(c)

Theorem 2 *Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R>0$ and let $f:(-R,R)\to\mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R,R)$ and $f'(x)=\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$.*

[4 marks; bookwork]

Note that the result from (b) plays an important role in the statement of the above theorem: it guarantees that the series $\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ appearing in the statement of this theorem converges. **[2 marks; bookwork]**

(d)

Corollary 1 *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $R > 0$ *and let* $f : (-R, R) \to \mathbb{R}$ *be the sum of this series. Then* f *is infinitely differentiable on the interval* $(-R, R)$ *and* $f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{a_n n!}{(n-k)!} x^{n-k}$.

[3 marks; bookwork]

Proof Repeated application of Theorem 2 and the result from (b). □ **[1 mark; bookwork]**

(e)(i) Fix an $x \neq 0$ and apply the ratio test: we have

$$\left| \frac{\frac{x^{3m+3}}{(2\cdot3)(5\cdot6)\cdots((3m-1)\cdot(3m))((3m+2)(3m+3)}}{\frac{x^{3m}}{(2\cdot3)(5\cdot6)\cdots((3m-1)\cdot(3m))}} \right| = \frac{|x|^3}{(3m+2)(3m+3)} \to 0$$

as $m \to \infty$. Hence, the power series converges absolutely for any x, so it has infinite radius of convergence. **[4 marks; unseen exercise]**

(e)(ii) The results from (d) and (e)(i) tell us that our function $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and that its derivatives are obtained by differentiating the power series for f term-by-term. Hence,

$$\begin{aligned} f''(x) = \sum_{m=1}^{\infty} \frac{(3m)(3m-1)x^{3m-2}}{(2\cdot3)(5\cdot6)\cdots((3m-1)\cdot(3m))} &= x + \sum_{m=2}^{\infty} \frac{(3m)(3m-1)x^{3m-2}}{(2\cdot3)(5\cdot6)\cdots((3m-1)\cdot(3m))} \\ &= x + x\sum_{m=2}^{\infty} \frac{x^{3m-3}}{(2\cdot3)(5\cdot6)\cdots((3m-4)\cdot(3m-3))} \,. \end{aligned}$$

Introducing a new summation index $p = m - 1$, we get

$$f''(x) = x + x\sum_{p=1}^{\infty} \frac{x^{3p}}{(2\cdot3)(5\cdot6)\cdots((3p-1)\cdot(3p))} = xf(x).$$

Thus, our function satisfies the differential equation $f'' - xf = 0$. **[6 marks; unseen exercise]**

Solution to question 5

(a) Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that

$$(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)\,.$$

[4 marks; bookwork]

(b) Fix $x > 0$ and consider the function

$$\begin{aligned} g(t) := f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - \dots \\ - \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x-t)^n\,, \quad (10) \end{aligned}$$

where $t \in (a,b)$ is the independent variable. Since $f:(a,b)\to\mathbb{R}$ is $n+1$ times differentiable, $g:(a,b)\to\mathbb{R}$ is differentiable. Differentiating (10) we get

$$\begin{aligned} g'(t) &= -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \dots \\ &\dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n \end{aligned}$$

(all the terms cancelled out, apart from the last one). Applying Cauchy's Generalisation of the Mean Value Theorem to the functions $g(t)$ and $h(t) := (x-t)^{n+1}$ on the interval $[0,x]$ (which we can because g is continuous on $[0,x]\subset(a,b)$ and differentiable on $(0,x)\subset(a,b)$) we get

$$\frac{g(x)-g(0)}{h(x)-h(0)} = \frac{g'(\xi)}{h'(\xi)} = \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{-(n+1)(x-\xi)^n} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi \in (0,x)$. But

$$g(0) = R_n(x), \quad g(x) = 0, \quad h(0) = x^{n+1}, \quad h(x) = 0,$$

so the above formula can be rewritten as

$$\frac{-R_n(x)}{-x^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,,$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$. **[11 marks; bookwork]**

(c) We have

$$\arctan' x = \frac{1}{1+x^2}\,, \qquad \arctan'' x = -\frac{2x}{(1+x^2)^2}\,, \qquad \arctan''' x = \frac{2(3x^2-1)}{(1+x^2)^3}\,,$$

so application of the result from (b) to the function $\arctan : \mathbb{R}\to\mathbb{R}$ with $n=2$ gives

$$\arctan x = x + \frac{3\xi^2-1}{3(1+\xi^2)^3}\,x^3 \tag{11}$$

for some $\xi \in (0,x)$. **[5 marks; unseen exercise]**

(d) Take $x = \frac{1}{\sqrt{3}}$. Then formula (11) becomes

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1-3\xi^2}{9\sqrt{3}(1+\xi^2)^3}$$

for some $\xi \in (0, \frac{1}{\sqrt{3}})$. The function $\xi \mapsto \frac{1-3\xi^2}{(1+\xi^2)^3}$ is strictly decreasing on the interval $[0,\frac{1}{\sqrt{3}}]$ (the numerator decreases as ξ increases, whereas the denominator increases as ξ increases), so

$$\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}\,,$$

or, equivalently, $\dfrac{16}{9}\sqrt{3} < \pi < 2\sqrt{3}$. **[5 marks; unseen exercise]**