All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Give, in terms of ε and δ, the mathematical definition of the statement "the function $f : [0, +\infty) \to \mathbb{R}$ is continuous at the point $x_0 \in [0, +\infty)$".
   (b) Give the mathematical definition of the statement "the function $f : [0, +\infty) \to \mathbb{R}$ is continuous".
   (c) Give, in terms of ε and δ, the mathematical definition of the statement "the function $f : [0, +\infty) \to \mathbb{R}$ is uniformly continuous".
   (d) Prove that the function $f : [0, +\infty) \to \mathbb{R}$, $f(x) = e^{-x}$, is uniformly continuous.
   (e) Prove that the function $f : [0, +\infty) \to \mathbb{R}$, $f(x) = e^{x}$, is not uniformly continuous.

2. (a) What is a partition of an interval $[a, b]$ and what is its mesh?
   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded.
      (i) Define the lower and upper Darboux sums of f with respect to a given partition of the interval $[a, b]$
      (ii) Define the lower and upper Riemann integrals of f over $[a, b]$.
      (iii) Give the mathematical definition of the statement "the function f is Riemann integrable on $[a, b]$". Define the Riemann integral $\displaystyle\int_a^b f(x)\,dx$.
      (iv) State Riemann's criterion for integrability.
   (c) Give the mathematical definition of the statement "the function $f : [a, b] \to \mathbb{R}$ is decreasing".
   (d) Prove that a decreasing function $f : [a, b] \to \mathbb{R}$ is bounded.
   (e) Prove that a decreasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. Here you may use without proof Riemann's criterion for integrability.
   (f) Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which is not continuous but is Riemann integrable on $[0, 1]$. Justify your answer.

3. (a) Let r be a natural number and let $f : [r, +\infty) \to [0, +\infty)$ be a decreasing function. Put

$$T_n := \sum_{k=r}^{n} f(k) - \int_r^n f(x)\,dx\,, \qquad n \in \mathbb{N}, \quad n \geq r.$$

   (i) Explain why the integral in the above formula makes sense, i.e. explain why the function f is Riemann integrable on $[r, n]$.
   (ii) Prove that the sequence $\{T_n\}$ is decreasing.
   (iii) Prove that $T_n \geq f(n)$.
   (iv) Prove that $T_n \geq 0$.
   (v) Prove that the sequence $\{T_n\}$ converges.
   (vi) Prove that the series $\sum\limits_{k=r}^{\infty} f(k)$ converges if and only if $\lim\limits_{n\to\infty} \int\limits_r^n f(x)\,dx$ exists.

(b) Does the series $\sum\limits_{k=2}^{\infty} \dfrac{1}{k \ln k}$ converge or diverge? Justify your answer.

4. (a) Define the notion of the radius of convergence of a real power series.

(b) Let k be a nonnegative integer. Consider two real power series,

$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n x^{n-k}\,.$$

How are their radii of convergence related? [No proof required.]

(c) State the theorem about the differentiability of a real power series. Explain how the result from (b) is used in the statement of this theorem.

(d) State and prove the corollary about the infinite differentiability of a real power series.

(e) Consider the real power series

$$1 + \sum_{m=1}^{\infty} \frac{x^{3m}}{(2\cdot 3)(5\cdot 6)\cdots((3m-1)\cdot(3m))}\,. \tag{1}$$

   (i) Prove that the power series (1) has infinite radius of convergence.
   (ii) Consider the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x)$ is given by formula (1). Prove that this function satisfies the differential equation $f'' - xf = 0$.

5. (a) State Cauchy's Generalisation of the Mean Value Theorem.

(b) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a, b) \to \mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

Given an $x \in (0, b)$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$ for some $\xi \in (0, x)$.

(c) Let $x > 0$. Prove that $\arctan x = x + \dfrac{3\xi^2 - 1}{3(1+\xi^2)^3}\,x^3$ for some $\xi \in (0, x)$.

(d) Prove that $\dfrac{16}{9}\sqrt{3} < \pi < 2\sqrt{3}$.

[*Hint:* apply the result from (c) with a particular x.]