MATH1102 Analysis

Solutions to examination paper for the academic year 2015/2016

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Solution to question 1

(a)(i)

Definition 1 *We say that* $L = \lim_{n\to\infty} x_n$ *if* $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N}$ *such that*

$$n > N \qquad \Longrightarrow \qquad |x_n - L| < \varepsilon.$$

[3 marks; bookwork]

(a)(ii)

Definition 2 *A sequence of real numbers* $\{x_n\}$ *is said to be a* Cauchy sequence *if* $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N}$ *such that*

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < \varepsilon.$$

[4 marks; bookwork]

(a)(iii)

Theorem 1 *A sequence of real numbers* $\{x_n\}$ *converges iff it is a Cauchy sequence.*

[3 marks; bookwork]

(b)

Definition 3 *We say that the function* $f : (0, 1] \to \mathbb{R}$ *is* uniformly continuous *if* $\forall \varepsilon > 0 \;\; \exists \delta > 0$ *such that*

$$|x - y| < \delta \quad \& \quad x, y \in (0, 1] \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon.$$

[4 marks; bookwork]

(c) Let ε be an arbitrary positive number. The definition of uniform continuity (Definition 3) tells us that there exists a $\delta > 0$ such that

$$|x - y| < \delta \quad \& \quad x, y \in (0, 1] \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon. \tag{1}$$

The sequence $\left\{\frac{1}{n}\right\}$ converges to 0, hence, by Cauchy's General Principle of Convergence (Theorem 1) it is a Cauchy sequence. The definition of a Cauchy sequence (Definition 2) tells us that there exists an $N \in \mathbb{N}$ such that

$$m, n > N \qquad \Longrightarrow \qquad \left|\frac{1}{m} - \frac{1}{n}\right| < \delta, \tag{2}$$

where the δ is the one from formula (1). Combining formulae (1) and (2) we get

$$m, n > N \qquad \Longrightarrow \qquad \left| f\left(\frac{1}{m}\right) - f\left(\frac{1}{n}\right) \right| < \varepsilon,$$

which in view of Definition 2, means that $\left\{f\left(\frac{1}{n}\right)\right\}$ is a Cauchy sequence. Cauchy's General Principle of Convergence (Theorem 1) now tells us that the sequence $\left\{f\left(\frac{1}{n}\right)\right\}$ converges. **[11 marks; similar to an exercise from an exercise sheet]**

Solution to question 2

(a)

Definition 4

(i) A partition *P of the interval $[a,b]$ is a finite sequence of real numbers $P = \{x_0, x_1, \ldots, x_n\}$ such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The set of all partitions of a given interval $[a,b]$ will be denoted by $\mathcal{P}[a,b]$.

(ii) The mesh (norm, width) *of the partition $P = \{x_0, x_1, \ldots, x_n\}$ is the number*

$$\|P\| := \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

[3 marks; bookwork]

(b)(i)

Definition 5 *Let $f \in \mathcal{B}[a,b]$, $P \in \mathcal{P}[a,b]$, $P = \{x_0, x_1, \ldots, x_n\}$. The* upper Darboux sum *of f with respect to the partition P is defined as*

$$U(f,P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i := \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$.

[3 marks; bookwork]

(b)(ii)

Definition 6 *Let $f \in \mathcal{B}[a,b]$. The* upper Riemann integral *of f over $[a,b]$ is defined as*

$$\overline{\int_a^b} f(x)\,dx := \inf_{P \in \mathcal{P}[a,b]} U(f,P).$$

[3 marks; bookwork]

(c)(i) We have $\sup\limits_{x \in [x_{i-1},\, x_i]} f(x) = x_i$, so

$$\left(\sup_{x \in [x_{i-1},\, x_i]} f(x) \right) (x_i - x_{i-1}) = x_i (x_i - x_{i-1}) = \frac{1}{2}(x_i^2 - x_{i-1}^2) + \frac{1}{2}(x_i - x_{i-1})^2. \tag{3}$$

Formula (3) implies

$$\frac{1}{2}(x_i^2 - x_{i-1}^2) < \left(\sup_{x \in [x_{i-1},\, x_i]} f(x) \right) (x_i - x_{i-1}) \le \frac{1}{2}(x_i^2 - x_{i-1}^2) + \frac{1}{2}(x_i - x_{i-1}) \|P\| \,. \tag{4}$$

[9 marks; unseen exercise]

(c)(ii) Summing up the inequalities (4) over $i = 1, 2, \ldots, n$, we get

$$\frac{1}{2} < U(f, P) \le \frac{1}{2}(1 + \|P\|). \tag{5}$$

[3 marks; unseen exercise]

(d) The left inequality (5) tells is that $\frac{1}{2}$ is a lower bound for upper Darboux sums, whereas the right inequality (5) tells us that $\frac{1}{2}$ is the greatest lower bound for upper Darboux sums (because the mesh can be chosen to be arbitrarily small). Hence, by the definition of the upper Riemann integral (Definition 6), $\overline{\int}_0^1 x\, dx = \frac{1}{2}$. **[4 marks; unseen exercise]**

Solution to question 3

(a)

Definition 7 *Let $f : I \to \mathbb{R}$. A* primitive *of f is a function $F : I \to \mathbb{R}$ which is continuous on I, differentiable on the interior of I and satisfies $F'(x) = f(x)$ on the interior of I.*

[2 marks; bookwork]

(b) No: one can always add a constant to F. **[1 mark; bookwork]**

(c)

Theorem 2 *Let $f \in \mathcal{R}[a, b]$ and let F be a primitive of f. Then*

$$\int_a^b f(x)\, dx \;=\; F(b) - F(a) \;=\; F(x)|_a^b \,.$$

[4 marks; bookwork]

Note: $F(x)|_a^b$ is simply a short way of writing $F(b) - F(a)$.

Proof of Theorem 2 Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Applying the Mean Value Theorem to the function F on the subinterval $[x_{i-1}, x_i]$ we get

$$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1})$$

for some $t_i \in (x_{i-1}, x_i)$. But

$$m_i \le f(t_i) \le M_i$$

where I use the standard notation $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$, $M_i := \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$. Hence,

$$m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1}).$$

Summing up over i from 1 to n we get

$$L(f, P) \le \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \le U(f, P).$$

But

$$\sum_{i=1}^{n}(F(x_i)-F(x_{i-1}))=F(x_n)-F(x_0)=F(b)-F(a),$$

so the above inequality can be rewritten as

$$L(f,P)\le F(b)-F(a)\le U(f,P). \tag{6}$$

The inequality (6) is true for any partition P. Taking a limiting sequence of partitions $\{P_n\}$ and letting $n\to\infty$ we get from (6)

$$\lim_{n\to\infty}L(f,P_n)\le F(b)-F(a)\le \lim_{n\to\infty}U(f,P_n).$$

It remains to note that as $f\in\mathcal{R}[a,b]$ the result stated in the examination paper tells us that

$$\lim_{n\to\infty}L(f,P_n)=\lim_{n\to\infty}U(f,P_n)=\int_a^b f(x)\,dx\,.$$

□ **[8 marks; bookwork]**

(d)

Definition 8 *Let $f\in\mathcal{R}_{loc}(I)$. An* indefinite integral *of f is a function $F:I\to\mathbb{R}$ defined by*

$$F(x)=\int_a^x f(t)\,dt$$

for some $a\in I$.

[2 marks; bookwork]

(e)

Theorem 3 *Let $f\in\mathcal{R}_{loc}(I)$ and let $F:I\to\mathbb{R}$ be an indefinite integral of f. Then*

(i) F is continuous on I;

(ii) F is differentiable at each interior point $x_0\in I$ at which f is continuous, and at such a point $F'(x_0)=f(x_0)$;

(iii) if f is continuous on I then F is a primitive of f.

[4 marks; bookwork]

(f) We have

$$G(x)=F(\cos x)-F(\sin x)$$

where

$$F(y)=\int_0^y e^{t^2}\,dt\,.$$

Hence, by the Chain Rule and the Theorem about the Properties of the Indefinite Integral we have

$$G'(x)=-F'(\cos x)\sin x-F'(\sin x)\cos x=-e^{\cos^2 x}\sin x-e^{\sin^2 x}\cos x.$$

[4 marks; unseen exercise]

Solution to question 4

(a) Let us introduce the set

$$S := \left\{ r \geq 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ converges for } x = r \text{ or } x = -r \right\}. \tag{7}$$

Obviously, $\{0\} \subset S \subset [0, +\infty)$.

Definition 9 *The* radius of convergence *of a power series is the extended real number*

$$R := \sup S$$

where S is the set (7).

[3 marks; bookwork]

(b) The fact that the power series diverges for $|x| > R$ follows from the definition of the radius of convergence R: indeed, the fact that $|x| > R$ and Definition 9 imply that $|x| \notin S$, S being the set (7). So we only need to prove absolute convergence for $|x| < R$.

Suppose $|x| < R$. Then, by the definition of R, there exists a y such that $|x| < |y| \leq R$ and $\sum_{n=0}^{\infty} a_n y^n$ converges. But convergence of the series $\sum_{n=0}^{\infty} a_n y^n$ implies that $\lim_{n\to\infty} a_n y^n = 0$, which in turn implies that the sequence $\{a_n y^n\}$ is bounded (a convergent sequence is bounded). Thus, there exists an $M \geq 0$ such that $|a_n y^n| \leq M$, $\forall n \in \mathbb{N} \cup \{0\}$. Hence,

$$|a_n x^n| = |a_n y^n| \left|\frac{x}{y}\right|^n \leq M \left|\frac{x}{y}\right|^n.$$

But

$$\sum_{n=0}^{\infty} M \left|\frac{x}{y}\right|^n < \infty$$

because this is a geometric series with ratio $\left|\frac{x}{y}\right| < 1$, so $\sum_{n=0}^{\infty} |a_n x^n|$ converges by the comparison test. $\square$ **[8 marks; bookwork]**

(c) The two power series, $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$, have the same radii of convergence. **[2 marks; bookwork]**

(d)

Theorem 4 *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.*

[4 marks; bookwork]

(e)(i) Fix an $x \neq 0$ and apply the ratio test: we have for $n \geq 1$

$$\left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = \frac{n - \frac{1}{2}}{n+1} |x| \to |x|$$

as $n \to \infty$. Hence, the power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$, so by the result from part (b) its radius of convergence is 1. **[3 marks; unseen exercise]**

(e)(ii) The results from (d) and (e)(i) tell us that the function f is differentiable on $(-1,1)$ and that its derivative is obtained by differentiating the power series for f term-by-term. Hence,

$$(1+x)f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^n = a_1 + \sum_{n=1}^{\infty} [na_n + (n+1)a_{n+1}]x^n. \tag{8}$$

We have

$$a_1 = \frac{1}{2}a_0\,, \tag{9}$$

and

$$a_{n+1} = \frac{\frac{1}{2}-n}{n+1}a_n\,, \quad \text{for} \quad n \geq 1. \tag{10}$$

Formula (10) can be equivalently rewritten as

$$na_n + (n+1)a_{n+1} = \frac{1}{2}a_n\,, \quad \text{for} \quad n \geq 1. \tag{11}$$

Formulae (8), (9) and (11) imply $(1+x)f'(x) = \frac{1}{2}f(x)$. **[5 marks; unseen exercise]**

Solution to question 5

(a)

Theorem 5 *Let* $f, g : (a,b) \to \mathbb{R}$ *be differentiable and let* $c \in (a,b)$ *be such that* $f(c) = g(c) = 0$ *and* $g'(x) \neq 0$ *for* $x \neq c$*. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{12}$$

provided the latter limit exists.

[5 marks; bookwork]

Explanation why $g(x) \neq 0$ for $x \neq c$ (this is not part of the proof of the theorem). Indeed, suppose $g(x) = 0$ for some $x \in (a,b)$, $x \neq c$.

Case 1: $x > c$. Apply Rolle's Theorem to the function g on the interval $[c,x]$: Rolle's Theorem tells us that there exists a $\xi \in (c,x)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 5.

Case 2: $x < c$. Apply Rolle's Theorem to the function g on the interval $[x,c]$: Rolle's Theorem tells us that there exists a $\xi \in (x,c)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 5.

Proof of Theorem 5 Suppose $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists and equals L. Then

$$\lim_{x\to c+} \frac{f'(x)}{g'(x)} = \lim_{x\to c-} \frac{f'(x)}{g'(x)} = L.$$

In order to prove (12) we need to show that

$$\lim_{x\to c+} \frac{f(x)}{g(x)} = L \tag{13}$$

and

$$\lim_{x\to c-} \frac{f(x)}{g(x)} = L. \tag{14}$$

Let us prove (13). Take an arbitrary sequence $\{x_n\} \subset (c,b)$ such that

$$\lim_{n\to\infty} x_n = c. \tag{15}$$

Applying Cauchy's Generalisation of the Mean Value Theorem on the interval $[c, x_n]$ we get a sequence $\{\xi_n\}$ such that

$$c < \xi_n < x_n \tag{16}$$

and

$$(g(x_n) - g(c))f'(\xi_n) = (f(x_n) - f(c))g'(\xi_n)$$

which can be equivalently rewritten as

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(\xi_n)}{g'(\xi_n)}\,. \tag{17}$$

Formulae (15), (16) and the Sandwich Principle imply

$$\lim_{n\to\infty} \xi_n = c. \tag{18}$$

Formula (18) and the sequential definition of a limit (one-sided version) imply

$$\lim_{n\to\infty} \frac{f'(\xi_n)}{g'(\xi_n)} = L. \tag{19}$$

Formulae (17) and (19) imply

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L. \tag{20}$$

Formula (20) and the sequential definition of a limit (one-sided version) imply (13).

Formula (14) is proved similarly. $\square$ **[9 marks; bookwork]**

(b)(i) Using L'Hôpital's Rule twice we get

$$\lim_{x\to-2} \frac{x^3+6x^2+12x+8}{x^3+5x^2+8x+4} = \lim_{x\to-2} \frac{3x^2+12x+12}{3x^2+10x+8} = \lim_{x\to-2} \frac{6x+12}{6x+10} = \frac{6\times(-2)+12}{6\times(-2)+10} = 0.$$

[3 marks; unseen exercise]

(b)(ii) Using L'Hôpital's Rule once we get

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)\tan x = \lim_{x\to\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin x + \left(\frac{\pi}{2} - x\right)\cos x}{-\sin x} = 1$$

so

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^2 \tan^2 x = \left(\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)\tan x\right)^2 = 1.$$

[3 marks; unseen exercise]

(b)(iii) We have

$$\begin{aligned}\frac{\tan^3 x - \sin^3 x}{x^5} &= \left(\frac{\sin x}{x}\right)^3 \frac{\frac{1}{\cos x} - 1}{x^2}\left(\frac{1}{\cos^2 x} + \frac{1}{\cos x} + 1\right) \\ &= \left(\frac{\sin x}{x}\right)^3 \frac{1-\cos x}{x^2}\left(\frac{1}{\cos^3 x} + \frac{1}{\cos^2 x} + \frac{1}{\cos x}\right). \end{aligned} \tag{21}$$

It is a standard fact (proved by applying L'Hôpital's Rule once) that

$$\lim_{x\to0} \frac{\sin x}{x} = 1. \tag{22}$$

By the continuity of the function cos we have

$$\lim_{x\to 0}\left(\frac{1}{\cos^3 x}+\frac{1}{\cos^2 x}+\frac{1}{\cos x}\right)=\frac{1}{\cos^3 0}+\frac{1}{\cos^2 0}+\frac{1}{\cos 0}=3. \tag{23}$$

Using L'Hôpital's Rule twice we get

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{\cos 0}{2}=\frac{1}{2}. \tag{24}$$

Combining formulae (21)–(24) we get

$$\lim_{x\to 0}\frac{\tan^3 x-\sin^3 x}{x^5}=\frac{3}{2}\,.$$

[5 marks; unseen exercise]

Solution to question 6

(a)

Definition 10 *A function* $f : I \to \mathbb{R}$ *is said to be* locally Riemann integrable *over* I *if* $\forall a, b \in I$ *we have* $f \in \mathcal{R}[a,b]$. *The set of all locally Riemann integrable functions over the interval* I *will be denoted by* $\mathcal{R}_{loc}(I)$.

[2 marks; bookwork]

(b)

Definition 11 *Let* $f \in \mathcal{R}_{loc}(I)$, $I = \langle c, d\rangle$.

(i) We say that f *is integrable at* c *in the improper sense if* $\exists p \in I$, $c < p$, *such that* $\lim_{a\to c+}\int_a^p f(x)\,dx$ *exists. We will denote this limit by* $\int_c^p f(x)\,dx\,.$

(ii) We say that f *is integrable at* d *in the improper sense if* $\exists p \in I$, $p < d$, *such that* $\lim_{b\to d-}\int_p^b f(x)\,dx$ *exists. We will denote this limit by* $\int_p^d f(x)\,dx\,.$

(iii) We say that f *is integrable over* I *in the improper sense if both (i) and (ii) hold for the same* p. *We will then write* $\int_c^d f(x)\,dx := \int_c^p f(x)\,dx + \int_p^d f(x)\,dx\,.$

[5 marks; bookwork]

(c)

Theorem 6 *Let* $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d\rangle$, *and suppose that*

- $0 \le f(x) \le g(x)$ *for all* $x \in I$;
- $\int_c^d g(x)\,dx$ *exists.*

Then $\int_c^d f(x)\,dx$ *exists.*

[3 marks; bookwork]

Proof Given arbitrary $a, b \in I$, $a < b$, we have

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx\,. \tag{25}$$

But according to the fact stated in the examination paper there exists a K such that

$$\int_a^b g(x)\,dx \le K \tag{26}$$

irrespective of the choice of a, b. Combining formulae (25) and (26) we get

$$\int_a^b f(x)\,dx \le K$$

irrespective of the choice of a, b. But according to the fact stated in the examination paper this implies that $\int_c^d f(x)\,dx$ exists. $\square$ **[5 marks; bookwork]**

(d) Introduce the functions

$$f^+(x) := \frac{|f(x)| + f(x)}{2}\,, \qquad f^-(x) := \frac{|f(x)| - f(x)}{2}\,.$$

It is easy to see that

$$0 \le f^+(x) \le |f(x)|, \tag{27}$$

$$0 \le f^-(x) \le |f(x)| \tag{28}$$

(to check the above inequalities consider separately the cases $f(x) \ge 0$ and $f(x) < 0$). By the Theorem about the Properties of the Riemann Integral we have $f^+, f^- \in \mathcal{R}_{loc}(I)$. Formula (27) and the Comparison Theorem for Improper Integrals (Theorem 6) imply that f^+ is integrable over I. Similarly, formula (28) implies that f^- is integrable over I. But $f(x) = f^+(x) - f^-(x)$, so the integrability of f over I follows by linearity. $\square$ **[4 marks; bookwork]**

(e) Let us prove the existence of the improper integral

$$\int_0^\infty \frac{\sin x}{1 + x^4}\,dx\,. \tag{29}$$

As there is no problem at $x = 0$, in order to prove the existence of the integral (29) it is sufficient to prove the existence of the integral

$$\int_1^\infty \frac{\sin x}{1 + x^4}\,dx\,. \tag{30}$$

By the result from (d), in order to prove the existence of the integral (30) it is sufficient to prove the existence of the integral

$$\int_1^\infty \frac{|\sin x|}{1 + x^4}\,dx\,. \tag{31}$$

Compare now the functions $\dfrac{|\sin x|}{1 + x^4}$ and $\dfrac{1}{x^4}$ on the interval $[1, \infty)$ We have

$$0 \le \frac{|\sin x|}{1 + x^4} \le \frac{1}{x^4}$$

so, by Theorem 6, in order to prove the existence of the integral (31) it is sufficient to prove the existence of the integral

$$\int_1^\infty \frac{1}{x^4}\,dx\,. \tag{32}$$

But the existence of the integral (32) is a standard fact. **[6 marks; unseen exercise]**