All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $\{x_n\}$ be a sequence of real numbers.
   (i) Give the mathematical definition of the statement "the sequence $\{x_n\}$ converges to a limit L".
   (ii) Give the mathematical definition of the statement "$\{x_n\}$ is a Cauchy sequence".
   (iii) State Cauchy's General Principle of Convergence for $\{x_n\}$.

   (b) Give the mathematical definition of the statement "the function $f : (0, 1] \to \mathbb{R}$ is uniformly continuous".

   (c) Let $f : (0, 1] \to \mathbb{R}$ be a uniformly continuous function. Prove that the sequence $\left\{f\left(\frac{1}{n}\right)\right\}$ converges.
   [*Hint:* you may find it helpful to prove that $\left\{f\left(\frac{1}{n}\right)\right\}$ is a Cauchy sequence.]

2. (a) What is a partition P of an interval $[a, b]$ and what is its mesh $\|P\|$?

   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded.
   (i) Define $U(f, P)$, the upper Darboux sum of f with respect to the given partition P of the interval $[a, b]$.
   (ii) Define $\overline{\int}_a^b f(x)\,dx$, the upper Riemann integral of f over $[a, b]$.

   (c) Consider the function $f : [0, 1] \to \mathbb{R}$, $f(x) = x$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[0, 1]$.
   (i) Prove that for all $i = 1, 2, \ldots, n$ we have
   $$\frac{1}{2}(x_i^2 - x_{i-1}^2) < \left(\sup_{x \in [x_{i-1},\, x_i]} f(x)\right)(x_i - x_{i-1}) \le \frac{1}{2}(x_i^2 - x_{i-1}^2) + \frac{1}{2}(x_i - x_{i-1})\|P\| \,.$$
   (ii) Prove that $\frac{1}{2} < U(f, P) \le \frac{1}{2}(1 + \|P\|)$.

   (d) Use the definition from (b)(ii) and the result from (c)(ii) to prove that $\overline{\int}_0^1 x\,dx = \frac{1}{2}$.

3. In this question $I \subset \mathbb{R}$ is an interval. The endpoints of this interval may be real numbers or $\pm\infty$.

   (a) Define what it means for a function $F : I \to \mathbb{R}$ to be a *primitive* of the function $f : I \to \mathbb{R}$.

   (b) Suppose that the function $f : I \to \mathbb{R}$ has a primitive $F : I \to \mathbb{R}$. Is this primitive unique?

   (c) State and prove the Fundamental Theorem of Calculus.

   Here you may use without proof the following fact: if the function f is Riemann integrable over $[a, b]$ and $\{P_n\}$ is a sequence of partitions such that $\lim_{n\to\infty} \|P_n\| = 0$, then $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \int_a^b f(x)\,dx$.

   (d) Let $I \subset \mathbb{R}$ be an interval. Define what it means for a function $F : I \to \mathbb{R}$ to be an indefinite integral of the locally Riemann integrable function $f : I \to \mathbb{R}$.

   (e) State the theorem about the properties of the indefinite integral.

   (f) Find the derivative of the function $G(x) = \displaystyle\int_{\sin x}^{\cos x} e^{t^2}\,dt$.

4. (a) Define the notion of the radius of convergence of a real power series.

   (b) Let R be the radius of convergence of a real power series. Prove that if $|x| < R$ then the power series converges absolutely and if $|x| > R$ then the power series diverges.

   (c) Consider two real power series, $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$. How are their radii of convergence related? [No proof required.]

   (d) State the theorem about the differentiability of real power series.

   (e) Consider the power series $\sum\limits_{n=0}^{\infty} a_n x^n$ where $a_0 = 1$ and

   $$a_n = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{3}{2} - n\right)}{n!} \quad \text{for} \quad n \in \mathbb{N}.$$

   (i) Find the radius of convergence.

   (ii) Prove that the function $f : (-1, 1) \to \mathbb{R}$, $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ satisfies the differential equation $(1 + x)f'(x) = \frac{1}{2} f(x)$.

5. (a) State and prove l'Hôpital's Rule for finding $\lim_{x \to c} \frac{f(x)}{g(x)}$ when $f(c) = g(c) = 0$.

   Here you may use without proof Cauchy's Generalisation of the Mean Value Theorem.

   (b) Evaluate the following limits:

   (i) $\lim_{x \to -2} \dfrac{x^3 + 6x^2 + 12x + 8}{x^3 + 5x^2 + 8x + 4}$,

   (ii) $\lim_{x \to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right)^2 \tan^2 x$,

   (iii) $\lim_{x \to 0} \dfrac{\tan^3 x - \sin^3 x}{x^5}$.

6. In this question $I \subset \mathbb{R}$ is an interval with endpoints c and d. These endpoints may be real numbers or $\pm\infty$.

   (a) Define what it means for a function $f : I \to \mathbb{R}$ to be *locally Riemann integrable.*

   (b) Let $f : I \to \mathbb{R}$ be locally Riemann integrable. Define what it means for f to be integrable over I in the improper sense.

   (c) State and prove the Comparison Theorem for Improper Integrals.

   You may use without proof the following fact. Let $f : I \to [0, +\infty)$ be a locally Riemann integrable function. Then f is integrable over I if and only if there exists a constant K such that for any $a, b \in I$, $a < b$, we have $\int_a^b f(x)\, dx \leq K$.

   (d) Suppose that f is locally Riemann integrable over I and suppose that $\int_c^d |f(x)|\, dx$ exists. Prove that $\int_c^d f(x)\, dx$ exists.

   (e) Prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{1 + x^4}\, dx$.

   [*Hint:* you may find it helpful to use the results from (c) and (d).]