MATH1102 Analysis

Solutions to examination paper for the academic year 2014/2015

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Solution to question 1

(a)

Definition 1 *We say that* $L = \lim_{n\to\infty} x_n$ *if* $\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N}$ *such that*

$$n > N \qquad \Longrightarrow \qquad |x_n - L| < \varepsilon.$$

[4 marks; bookwork]

(b)

Definition 2 *A sequence of real numbers* $\{x_n\}$ *is said to be a* Cauchy sequence *if* $\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N}$ *such that*

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < \varepsilon.$$

[4 marks; bookwork]

(c) Suppose that the sequence $\{x_n\}$ is a Cauchy sequence. We need to prove that the sequence $\{x_n\}$ converges.

Set $\varepsilon = 1$ and apply Definition 2. We conclude that $\exists N \in \mathbb{N}$ such that

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < 1.$$

In particular,

$$n > N \qquad \Longrightarrow \qquad |x_n - x_{N+1}| < 1. \tag{1}$$

But

$$|x_n| = |(x_n - x_{N+1}) + x_{N+1}| \le |x_n - x_{N+1}| + |x_{N+1}|. \tag{2}$$

Combining formulae (1) and (2) we get

$$n > N \qquad \Longrightarrow \qquad |x_n| < |x_{N+1}| + 1.$$

Hence,

$$|x_n| \le \max\{|x_1|, \dots, |x_N|, |x_{N+1}| + 1\}$$

for all $n \in \mathbb{N}$, which means that the sequence $\{x_n\}$ is bounded.

According to the Bolzano–Weierstrass Theorem the sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ which converges to a limit L. We will now prove that that this L is the limit of the whole sequence $\{x_n\}$.

Let ε be an arbitrary positive number. Then by Definition 1 $\exists K \in \mathbb{N}$ such that

$$k > K \qquad \Longrightarrow \qquad |x_{n_k} - L| < \frac{\varepsilon}{2}.$$

Also, by Definition 2 $\exists M \in \mathbb{N}$ such that

$$m, n > M \qquad \Longrightarrow \qquad |x_m - x_n| < \frac{\varepsilon}{2}.$$

Let us now **fix** a number $k \in \mathbb{N}$ such that $k > K$ and $n_k > M$. Then for $n > M$ we have

$$|x_n - L| = |(x_n - x_{n_k}) + (x_{n_k} - L)| \leq |x_n - x_{n_k}| + |x_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

[17 marks; bookwork]
[Note that this is the first time this question appears in an Analysis 2 examination paper]

Solution to question 2

(a)(i) The *lower Darboux sum* of f with respect to the partition $P = \{x_0, x_1, \ldots, x_n\}$ is defined as

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}) ,$$

where $m_i := \inf_{x \in [x_{i-1}, x_i]} f(x)$. The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}) ,$$

where $M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$. **[2 marks; bookwork]**

(a)(ii)

Definition 3 *Let $f \in \mathcal{B}[a, b]$. The* lower Riemann integral *of f over $[a, b]$ is defined as*

$$\underline{\int_a^b} f(x)\, dx := \sup_{P \in \mathcal{P}[a,b]} L(f, P) .$$

The upper Riemann integral *of f over $[a, b]$ is defined as*

$$\overline{\int_a^b} f(x)\, dx := \inf_{P \in \mathcal{P}[a,b]} U(f, P) .$$

[2 marks; bookwork]

(a)(iii)

Definition 4 *The function $f \in \mathcal{B}[a, b]$ is said to be* Riemann integrable *on $[a, b]$ if*

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx .$$

In this case the common value of the lower and upper Riemann integrals is denoted by

$$\int_a^b f(x)\, dx$$

and is called the Riemann integral *of f over $[a, b]$. The set of all Riemann integrable functions $f : [a, b] \to \mathbb{R}$ will be denoted by $\mathcal{R}[a, b]$.*

[2 marks; bookwork]

(a)(iv) Suppose $\forall \varepsilon > 0 \ \exists P \in \mathcal{P}[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon. \tag{3}$$

holds. Need to prove that $f \in \mathcal{R}[a,b]$.

Let ε be an arbitrary positive number. Choose $P \in \mathcal{P}[a,b]$ such that (3) holds. Then

$$U(f,P) - \varepsilon < L(f,P) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(f,P),$$

so

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, this implies

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0$$

which means that $f \in \mathcal{R}[a,b]$. **[5 marks; bookwork]**

(a)(v)

Definition 5 *We say that the function* $f : [a,b] \to \mathbb{R}$ *is* uniformly continuous *if* $\forall \varepsilon > 0 \quad \exists \delta > 0$ *such that*

$$|x-y| < \delta \quad \& \quad x,y \in [a,b] \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon.$$

[2 marks; bookwork]

(a)(vi) Consider a function $f \in C[a,b]$. Let ε be an arbitrary positive number. Then, by the fact that a continuous function $f : [a,b] \to \mathbb{R}$ is uniformly continuous and Definition 5, $\exists \delta > 0$ such that

$$|x-y| < \delta \quad \& \quad x,y \in [a,b] \qquad \Longrightarrow \qquad |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}.$$

Choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ with $\|P\| < \delta$. Then

$$M_i - m_i \le \frac{\varepsilon}{2(b-a)};$$

here we use the usual notation $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$, $M_i := \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$. We get

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \le \frac{\varepsilon}{2(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\varepsilon}{2} < \varepsilon$$

and the result from (a)(iv) tells us that $f \in \mathcal{R}[a,b]$. **[6 marks; bookwork]**

(b) Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e & \text{if} \quad x = 0, \\ \sum\limits_{n=0}^{[x^{-1}]} \frac{1}{n!} & \text{if} \quad x > 0, \end{cases}$$

where $\left[x^{-1}\right]$ is the integer part of x^{-1}. Our function is decreasing on $[0,1]$, hence it is integrable on $[0,1]$.

Now, take an $n \in \mathbb{N}$, $n \geq 2$, and set $x_0 = \frac{1}{n}$. Then

$$\lim_{x \to x_0^-} f(x) - \lim_{x \to x_0^+} f(x) = \frac{1}{n!}\,,$$

so f is not continuous at x_0. We have established that our function has an infinite number of discontinuities. **[6 marks; unseen exercise]**

Solution to question 3

(a)

Theorem 1 *Let $f \in \mathcal{B}[a,b]$. Then $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that $\|P\| < \delta$ implies*

$$\underline{\int_a^b} f(x)\,dx\, - \varepsilon < L(f,P) \leq \underline{\int_a^b} f(x)\,dx\,, \tag{4}$$

$$\overline{\int_a^b} f(x)\,dx \leq U(f,P) < \overline{\int_a^b} f(x)\,dx\, + \varepsilon. \tag{5}$$

[4 marks; bookwork]

(b) Let ε be an arbitrary positive number. Darboux's Theorem states that $\exists \delta > 0$ such that $\|P\| < \delta$ implies

$$\underline{\int_a^b} f(x)\,dx\, - \varepsilon < L(f,P) \leq \underline{\int_a^b} f(x)\,dx\,,$$

$$\overline{\int_a^b} f(x)\,dx \leq U(f,P) < \overline{\int_a^b} f(x)\,dx\, + \varepsilon.$$

We have $\lim_{n\to\infty} \|P_n\| = 0$ so, by the definition of a limit of sequence, $\exists N \in \mathbb{N}$ such that $n > N$ implies $\|P_n\| < \delta$. Hence, for $n > N$ we have

$$\underline{\int_a^b} f(x)\,dx\, - \varepsilon < L(f,P_n) \leq \underline{\int_a^b} f(x)\,dx\,,$$

$$\overline{\int_a^b} f(x)\,dx \leq U(f,P_n) < \overline{\int_a^b} f(x)\,dx\, + \varepsilon.$$

By the definition of a limit of sequence this means that

$$\lim_{n\to\infty} L(f,P_n) = \underline{\int_a^b} f(x)\,dx\,, \qquad \lim_{n\to\infty} U(f,P_n) = \overline{\int_a^b} f(x)\,dx\,.$$

[7 marks; this question appeared in an exercise sheet]

(c)(i) Let $P_n = \{x_0, \ldots, x_n\}$ be the partition of $[a,b]$ with $x_i = a + \frac{i(b-a)}{n}$, $i = 0, \ldots, n$. For x in the interval $[x_{i-1}, x_i]$ we have $m_i \leq f(x) \leq M_i$ where we use the notation

$$m_i := \inf_{x\in[x_{i-1},x_i]} f(x), \qquad M_i := \sup_{x\in[x_{i-1},x_i]} f(x).$$

In particular,

$$m_i \leq f(x_i) \leq M_i.$$

Multiplying these inequalities by $\frac{b-a}{n}$ and summing over $i = 1, \dots, n$ we get

$$L(f, P_n) \le R(f, P_n) \le U(f, P_n). \tag{6}$$

[4 marks; unseen exercise]

(c)(ii) As $\{P_n\}$ is a limiting sequence of partitions and as f is Riemann integrable we have by the result from (b)

$$\lim_{n\to\infty} L(f, P_n) \;=\; \lim_{n\to\infty} U(f, P_n) \;=\; \int_a^b f(x)\, dx\,. \tag{7}$$

By the Sandwich Theorem we obtain from (6) and (7)

$$\lim_{n\to\infty} R(f, P_n) \;=\; \int_a^b f(x)\, dx\,.$$

[4 marks; unseen exercise]

(d) The function $f(x) = x$ is continuous on the whole real line, hence, Riemann integrable on any closed bounded interval. We have

$$R(f, P_n) = \frac{1}{n}\left[\sum_{i=1}^{n} \frac{i}{n}\right] = \frac{1}{n^2}\left[\sum_{i=1}^{n} i\right] = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}\left[1 + \frac{1}{n}\right].$$

Therefore, by (c)(ii)

$$\int_0^1 x\, dx = \lim_{n\to\infty} R(f, P_n) = \frac{1}{2}\,.$$

[6 marks; unseen exercise]

Solution to question 4

(a) Put $h(x) := f(x) - g(x)$. Then $h'(x) = 0$ for all $x \in (-1, 1)$. I claim that

$$h(x) = h(0), \qquad \forall x \in (-1, 1). \tag{8}$$

Case 1: $x = 0$. Obviously, (8) is true.

Case 2: $x \in (0, 1)$. Applying the Mean Value Theorem to the function h on the interval $[0, x]$ we get

$$\frac{h(x) - h(0)}{x - 0} = h'(\xi) \tag{9}$$

for some $\xi \in (0, x)$. But $h'(\xi) = 0$, so (9) implies (8).

Case 3: $x \in (-1, 0)$. Applying the Mean Value Theorem to the function h on the interval $[x, 0]$ we get

$$\frac{h(0) - h(x)}{0 - x} = h'(\xi) \tag{10}$$

for some $\xi \in (x, 0)$. But $h'(\xi) = 0$, so (10) implies (8). **[8 marks; unseen exercise, but somewhat similar to a lemma proved in the lectures and an exercise from an exercise sheet]**

(b) In the lectures the radius of convergence was defined as follows. Introduce the set

$$S := \left\{ r \ge 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ converges for } x = r \text{ or } x = -r \right\}. \tag{11}$$

Obviously, $\{0\} \subset S \subset [0, +\infty)$. The *radius of convergence* of a power series is the extended real number

$$R := \sup S$$

where S is the set (11).

Another possible definition is that the radius of convergence of a power series is the extended real number R such that for $|x| < R$ the series converges absolutely and for $|x| > R$ the series diverges. I will accept this definition even though it is not *a priori* obvious that an R with these properties exists. **[3 marks; bookwork]**

(c) Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$. **[4 marks; bookwork]**

(d) Fix an $x \neq 0$ and apply the ratio test:

$$\left| \frac{\frac{(-1)^{k+1}}{2k+3} x^{2k+3}}{\frac{(-1)^k}{2k+1} x^{2k+1}} \right| = \frac{2k+1}{2k+3} x^2 = \frac{x^2}{1 + \frac{2}{2k+1}} \to x^2$$

as $k \to \infty$. Hence, the power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is 1. **[4 marks; unseen exercise]**

(e) The results from (c) and (d) tell us that the function g is differentiable on $(-1, 1)$ and that its derivative is obtained by differentiating the power series for g term-by-term:

$$g'(x) = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \right)' = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1} x^{2k+1} \right)' = \sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2}$$

because we are looking at a geometric series. **[4 marks; unseen exercise]**

(f) Application of the result from (a) with $f(x) = \arctan x$ and $g(x)$ as in (e) gives

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + c, \qquad \forall x \in (-1, 1).$$

Setting $x = 0$ we conclude that $c = 0$. **[2 marks; unseen exercise]**

Solution to question 5

(a)

Theorem 2 *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $\xi \in (a, b)$ such that*

$$(g(b) - g(a)) f'(\xi) = (f(b) - f(a)) g'(\xi). \tag{12}$$

[4 marks; bookwork]

(b) Fix $x \in (a, 0)$ and consider the function

$$g(t) := f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - \dots - \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x-t)^n, \tag{13}$$

where t is the independent variable. Since $f:(a,b)\to\mathbb{R}$ is $n+1$ times differentiable, $g:(a,b)\to\mathbb{R}$ is differentiable. Differentiating (13) we get

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \dots$$
$$\dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right]$$
$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

(all the terms cancelled out, apart from the last one). Applying Cauchy's Generalisation of the Mean Value Theorem to the functions $g(t)$ and $h(t):=(x-t)^{n+1}$ on the interval $[x,0]$ (which we can because g is continuous on $[x,0]\subset(a,b)$ and differentiable on $(x,0)\subset(a,b)$) we get

$$\frac{g(0)-g(x)}{h(0)-h(x)} = \frac{g'(\xi)}{h'(\xi)} = \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{-(n+1)(x-\xi)^n} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi\in(x,0)$. But

$$g(0)=R_n(x), \quad g(x)=0, \quad h(0)=x^{n+1}, \quad h(x)=0,$$

so the above formula can be rewritten as

$$\frac{R_n(x)}{x^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$. $\square$ **[12 marks; bookwork; however in the lectures the above argument was done only for the case** $x>0$**]**

(c) Application of the result from (b) to the function $f:(-1,99)\to\mathbb{R}$, $f(x)=\ln(1+x)$ with $x=-\frac{1}{2}$ gives

$$\ln\left(\frac{1}{2}\right) = -\sum_{k=1}^{n}\frac{1}{k2^k} + R_n\left(-\frac{1}{2}\right),$$

$$R_n\left(-\frac{1}{2}\right) = (-1)^n\frac{\left(-\frac{1}{2}\right)^{n+1}}{(n+1)(1+\xi)^{n+1}} = -\frac{1}{(n+1)(2+2\xi)^{n+1}}$$

for some $\xi\in(-1/2,0)$. But the fact that $\xi>-1/2$ implies that $2+2\xi>1$ so

$$\left|R_n\left(-\frac{1}{2}\right)\right| = \frac{1}{(n+1)(2+2\xi)^{n+1}} < \frac{1}{(n+1)} \to 0 \qquad \text{as} \qquad n\to\infty$$

so

$$R_n\left(-\frac{1}{2}\right)\to 0 \qquad \text{as} \qquad n\to\infty$$

which gives the required result. **[9 marks; this question is a special case of a question from an exercise sheet; the tricky bit is that I advised students to use Cauchy's form of the remainder when expanding** $\ln(1+x)$ **for** $x<0$ **whereas in this examination question I ask them to use Lagrange's form of the remainder; as shown above, Lagrange's form of the remainder is actually OK for** $x\geq -1/2$**]**

Solution to question 6

(a)

Definition 6 *A function* $f : (0,1] \to \mathbb{R}$ *is said to be* locally Riemann integrable *over* $(0,1]$ *if* $\forall a, b \in (0,1]$ *we have* $f \in \mathcal{R}[a,b]$. *The set of all locally Riemann integrable functions over the interval* $(0,1]$ *will be denoted by* $\mathcal{R}_{loc}(0,1]$.

[2 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval I **and not specifically for the interval** $(0,1]$**]**

(b) We say that f is integrable on $(0,1]$ in the improper sense if $\lim\limits_{a\to 0^+} \int_a^1 f(x)\,dx$ exists. We will denote this limit by $\int_0^1 f(x)\,dx$. **[3 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval** I **and not specifically for the interval** $(0,1]$**]**

(c) Note that the integrand is continuous on $(0,1]$, hence locally integrable on $(0,1]$. So we have to check only integrability at 0. We have

$$\int_a^1 \ln x\,dx \;=\; x(\ln x - 1)\big|_a^1 = -1 - a(\ln a - 1) \to -1 \qquad \text{as} \qquad a \to 0^+.$$

[2 marks; special case of an argument done in the lectures]

(d)

Theorem 3 *Let* $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d\rangle$, *and suppose that*

- $0 \le f(x) \le g(x)$ *for all* $x \in I$*;*
- $\int_c^d g(x)\,dx$ *exists.*

Then $\int_c^d f(x)\,dx$ *exists.*

[3 marks; bookwork]

(e)

Theorem 4 *Suppose that* $f \in \mathcal{R}_{loc}(I)$ *and* $|f|$ *is integrable over* I. *Then* f *is integrable over* I.

[3 marks; bookwork]

(f) Let us prove the existence of the improper integral

$$\int_0^1 (\ln x) \cos\frac{1}{x}\,dx\,. \tag{14}$$

Note that the integrand is continuous on $(0,1]$, hence locally integrable on $(0,1]$. So we have to check only integrability at 0.

By Theorem 4, in order to prove the existence of the integral (14) it is sufficient to prove the existence of the integral

$$\int_0^1 |\ln x| \left|\cos\frac{1}{x}\right| dx\,. \tag{15}$$

Compare now the functions $|\ln x|\left|\cos\frac{1}{x}\right|$ and $-\ln x$ on the interval $(0,1]$. We have

$$0 \le |\ln x|\left|\cos\frac{1}{x}\right| \le -\ln x\,,$$

so, by Theorem 3, in order to prove the existence of the integral (15) it is sufficient to prove the existence of the integral $\int_0^1(-\ln x)\,dx$. But this has already been done in (c). [**5 marks; unseen exercise**]

(g) Note that the integrand is continuous on $(0,1]$, hence locally integrable on $(0,1]$. So we have to check only integrability at 0.

Integrating by parts we get

$$\int_a^1 \frac{1-\ln x}{x}\sin\frac{1}{x}\,dx \,=\, x(1-\ln x)\cos\frac{1}{x}\bigg|_a^1 + \int_a^1 (\ln x)\cos\frac{1}{x}\,dx = \cos 1 - a(1-\ln a)\cos\frac{1}{a} + \int_a^1 (\ln x)\cos\frac{1}{x}\,dx\,.$$

We have $\cos 1 - a(1-\ln a)\cos\frac{1}{a} \to \cos 1$ as $a \to 0^+$, so in order to prove the existence of the original integral we have to prove the existence of the integral $\int_0^1(\ln x)\cos\frac{1}{x}\,dx$. But this has already been done in (f). [**5 marks; unseen exercise**]

(h)

$$\int_0^1 \frac{1-\ln x}{x}\sin\frac{1}{x}\,dx \,=\, \int_0^1 (\ln x)\cos\frac{1}{x}\,dx \,+\cos 1\,.$$

[**2 marks; unseen exercise**]