All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let $\{x_n\}$ be a sequence of real numbers.
   (a) Define the statement "the sequence $\{x_n\}$ converges".
   (b) Define the statement "$\{x_n\}$ is a Cauchy sequence".
   (c) Prove that a Cauchy sequence converges. Here you may use without proof the Bolzano–Weierstrass Theorem.

2. (a) Let $f : [a, b] \to \mathbb{R}$ be bounded.
   (i) Given a partition P of the interval $[a, b]$, define the lower Darboux sum $L(f, P)$ and the upper Darboux sum $U(f, P)$.
   (ii) Define the lower and upper Riemann integrals of f over $[a, b]$.
   (iii) Define what it means for f to be Riemann integrable on $[a, b]$.
   (iv) Suppose that for any $\epsilon > 0$ there exists a partition P such that

   $$U(f, P) - L(f, P) < \epsilon.$$

   Prove that f is Riemann integrable on $[a, b]$. Here you may use without proof the fact that the lower Riemann integral is less than or equal to the upper Riemann integral.
   (v) Define the statement "the function $f : [a, b] \to \mathbb{R}$ is uniformly continuous".
   (vi) Let $f : [a, b] \to \mathbb{R}$ be continuous. Using the result from (iv), prove that f is Riemann integrable on $[a, b]$. Here you may use without proof the fact that if $f : [a, b] \to \mathbb{R}$ is continuous then it is uniformly continuous.

   (b) Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which has an infinite number of discontinuities but is Riemann integrable on $[0, 1]$. Briefly justify your answer. You may use any result from the course.

3. (a) State Darboux's Theorem from the theory of Riemann integration.

   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded and let $\{P_n\}$ be a limiting sequence of partitions of the interval $[a, b]$, i.e. sequence such that $\lim_{n\to\infty} \|P_n\| = 0$. Using Darboux's Theorem, prove that

$$\lim_{n\to\infty} L(f, P_n) = \underline{\int_a^b} f(x)\, dx\,, \qquad \lim_{n\to\infty} U(f, P_n) = \overline{\int_a^b} f(x)\, dx\,.$$

   (c) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and let $P_n = \{x_0, x_1 \dots, x_n\}$ be the partition of $[a, b]$ with $x_i = a + \frac{i(b-a)}{n}$, $i = 0, \dots, n$. Put

$$R(f, P_n) := \frac{b-a}{n} \sum_{i=1}^{n} f(x_i).$$

   (i) Show that $L(f, P_n) \le R(f, P_n) \le U(f, P_n)$.

   (ii) Using the result from (b), deduce that $\lim_{n\to\infty} R(f, P_n) = \int_a^b f(x)\, dx\,.$

   (d) Use the result from (c)(ii) to show that $\int_0^1 x\, dx = \frac{1}{2}\,.$

4. (a) Suppose that the functions $f, g : (-1, 1) \to \mathbb{R}$ are differentiable and that $f'(x) = g'(x)$ for all $x \in (-1, 1)$. Prove that $f(x) = g(x)+c$ for all $x \in (-1, 1)$, where c is a constant.

   (b) Define the notion of the radius of convergence of a power series.

   (c) State the theorem about the differentiability of power series.

   (d) Find the radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}\,.$

   (e) Consider the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}\,.$ Prove that

$$g'(x) = \frac{1}{1+x^2}\,.$$

   (f) Prove that $\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ for all $x \in (-1, 1)$. Here you may use without proof the fact that $\arctan' x = \frac{1}{1+x^2}\,.$

   [*Hint:* you may find it helpful to use the results from (a) and (e).]

5. (a) State Cauchy's Generalisation of the Mean Value Theorem.

   (b) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a, b) \to \mathbb{R}$ be $n + 1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

   Given an $x \in (a, 0)$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ for some $\xi \in (x, 0)$.

   (c) Use the result from (b) to prove that $\ln\left(\dfrac{1}{2}\right) = -\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

6. (a) Define what it means for a function $f : (0, 1] \to \mathbb{R}$ to be locally Riemann integrable.

   (b) Let $f : (0, 1] \to \mathbb{R}$ be locally Riemann integrable. Define what it means for f to be integrable on $(0, 1]$ in the improper sense.

   (c) Use the definition from (b) to prove the existence of the improper integral $\displaystyle\int_0^1 \ln x\, dx$ and to evaluate this integral.

   (d) State the Comparison Theorem for Improper Integrals.

   (e) State the theorem relating the integrability, in the improper sense, of the functions f and $|f|$.

   (f) Prove the existence of the improper integral $\displaystyle\int_0^1 (\ln x) \cos\frac{1}{x}\, dx$.

   [*Hint:* you may find it helpful to use the results from (c), (d) and (e).]

   (g) Prove the existence of the improper integral $\displaystyle\int_0^1 \frac{1 - \ln x}{x} \sin\frac{1}{x}\, dx$.

   [*Hint:* you may find it helpful to integrate by parts.]

   (h) How are the numerical values of the improper integrals from (f) and (g) related?