# MATH1102 Analysis

# Solutions to examination paper for the academic year 2013/2014

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**Solution to question 1**

**(a)**

**Definition 1** *We say that the function* $f : [a, b] \to \mathbb{R}$ *is* continuous at the point $x_0 \in [a, b]$ *if* $\forall \varepsilon > 0 \;\; \exists \delta > 0$ *such that*

$$|x - x_0| < \delta \quad \& \quad x \in [a, b] \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

**[4 marks; bookwork]**

**(b)**

**Definition 2** *We say that the function* $f : [a, b] \to \mathbb{R}$ *is* continuous *if it is continuous at every point* $x_0 \in [a, b]$.

**[3 marks; bookwork]**

**(c)**

**Definition 3** *We say that the function* $f : [a, b] \to \mathbb{R}$ *is* uniformly continuous *if* $\forall \varepsilon > 0 \;\; \exists \delta > 0$ *such that*

$$|x - y| < \delta \quad \& \quad x, y \in [a, b] \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon.$$

**[4 marks; bookwork]**

**(d)** Suppose $f$ is not uniformly continuous. Then $\exists \varepsilon > 0$ such that $\forall \delta > 0$ $\exists x, y \in [a, b]$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

Put $\delta = \frac{1}{n}$, $n = 1, 2, \ldots$. Then for each $n$ we get $x_n, y_n \in [a, b]$ such that

$$|x_n - y_n| < \frac{1}{n} \tag{1}$$

and

$$|f(x_n) - f(y_n)| \geq \varepsilon. \tag{2}$$

Since $x_n \in [a, b]$, the sequence $\{x_n\}$ is bounded. By the Bolzano–Weierstrass Theorem the sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ which converges to some $c$. This $c$ has to be in $[a, b]$ because if it were not then $|x_{n_k} - c|$ would be bounded from below by $a - c$ or $c - b$ (depending on whether $c < a$ or $c > b$), contradicting the fact that $\{x_{n_k}\}$ converges to $c$. Formula (1) implies

$$|y_{n_k} - c| = |(y_{n_k} - x_{n_k}) + (x_{n_k} - c)| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| \leq \frac{1}{n_k} + |x_{n_k} - c| \to 0$$

as $k \to \infty$, so the subsequence $\{y_{n_k}\}$ also converges to $c$.

Now, $f$ is continuous at $c$, so by the sequential definition of continuity

$$\lim_{k\to\infty} f(x_{n_k}) = f(c), \qquad \lim_{k\to\infty} f(y_{n_k}) = f(c).$$

By the definition of a limit of a sequence $\exists N, M \in \mathbb{N}$ such that

$$k > N \qquad \Longrightarrow \qquad |f(x_{n_k}) - f(c)| < \frac{\varepsilon}{2},$$

$$k > M \qquad \Longrightarrow \qquad |f(y_{n_k}) - f(c)| < \frac{\varepsilon}{2}.$$

Choose a $k$ such that $k > N$ and $k > M$. Then

$$|f(x_{n_k}) - f(y_{n_k})| = |(f(x_{n_k}) - f(c)) + (f(c) - f(y_{n_k}))| \leq$$
$$|f(x_{n_k}) - f(c)| + |f(c) - f(y_{n_k})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

contradicting (2). **[14 marks; bookwork]**

## Solution to question 2

**(a)**

**Definition 4**

*(i) A* partition *$P$ of the interval $[a, b]$ is a finite sequence of real numbers $P = \{x_0, x_1, \ldots, x_n\}$ such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

*The set of all partitions of a given interval $[a, b]$ will be denoted by $\mathcal{P}[a, b]$.*

*(ii) The* mesh (norm, width) *of the partition $P = \{x_0, x_1, \ldots, x_n\}$ is the number*

$$\|P\| := \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

**[2 marks; bookwork]**

**(b)(i)** The *lower Darboux sum* of $f$ with respect to the partition $P = \{x_0, x_1, \ldots, x_n\}$ is defined as

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

where $m_i := \inf_{x\in[x_{i-1},x_i]} f(x)$. The *upper Darboux sum* of $f$ with respect to the partition $P$ is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i := \sup_{x\in[x_{i-1},x_i]} f(x)$. **[3 marks; bookwork]**

**(b)(ii)**

**Definition 5** *Let* $f \in \mathcal{B}[a,b]$*. The* lower Riemann integral *of* $f$ *over* $[a,b]$ *is defined as*

$$\underline{\int_a^b} f(x)\,dx := \sup_{P \in \mathcal{P}[a,b]} L(f,P)\,.$$

*The* upper Riemann integral *of* $f$ *over* $[a,b]$ *is defined as*

$$\overline{\int_a^b} f(x)\,dx := \inf_{P \in \mathcal{P}[a,b]} U(f,P)\,.$$

**[3 marks; bookwork]**

**(b)(iii)**

**Definition 6** *The function* $f \in \mathcal{B}[a,b]$ *is said to be* Riemann integrable *on* $[a,b]$ *if*

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx\,.$$

*In this case the common value of the lower and upper Riemann integrals is denoted by*

$$\int_a^b f(x)\,dx$$

*and is called the* Riemann integral *of* $f$ *over* $[a,b]$*. The set of all Riemann integrable functions* $f : [a,b] \to \mathbb{R}$ *will be denoted by* $\mathcal{R}[a,b]$*.*

**[2 marks; bookwork]**

**(b)(iv)**

**Theorem 1** *Let* $f \in \mathcal{B}[a,b]$*. Then* $f \in \mathcal{R}[a,b]$ *iff* $\forall \varepsilon > 0 \ \ \exists P \in \mathcal{P}[a,b]$ *such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{3}$$

**[3 marks; bookwork]**

**(c)**

**Definition 7** *A function* $f : [a,b] \to \mathbb{R}$ *is said to be* decreasing *on* $[a,b]$ *if*

$$x_1 < x_2 \quad \& \quad x_1, x_2 \in [a,b] \qquad \Longrightarrow \qquad f(x_1) \geq f(x_2).$$

**[1 mark; bookwork]**

**(d)** Consider a partition $P_n$ into $n$ subintervals of equal length: $P_n = \{x_0, x_1, \ldots, x_n\}$,

$$x_i = a + \frac{b-a}{n} i, \qquad i = 0, 1, \ldots, n.$$

Then

$$L(f,P_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i), \qquad U(f,P_n) = \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i),$$

$$U(f,P_n) - L(f,P_n) = \frac{b-a}{n}(f(x_0) - f(x_n)) = \frac{b-a}{n}(f(a) - f(b))\,.$$

Now, let $\varepsilon$ be an arbitrary positive number. Choose $n$ so large that

$$\frac{b-a}{n}(f(a)-f(b)) < \varepsilon.$$

Then

$$U(f,P_n) - L(f,P_n) < \varepsilon$$

and Riemann's Criterion for Integrability tells us that $f$ is Riemann integrable on $[a,b]$. **[7 marks; for the case of increasing function proof done in lectures]**

**(e)** Our function is decreasing on $[0,1]$, so by the result from (d) it is integrable on $[0,1]$. **[4 marks; unseen exercise]**

## Solution to question 3

**(a)(i)** Put

$$T_n := \sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx \tag{4}$$

for $n \in \mathbb{N}$, $n \geq 1$.

We have

$$T_n - T_{n+1} = \int_n^{n+1} f(x)\,dx - f(n+1) = \int_n^{n+1} f(x)\,dx - \int_n^{n+1} f(n+1)\,dx$$
$$= \int_n^{n+1} (f(x) - f(n+1))\,dx \geq 0,$$

so the sequence $\{T_n\}$ is decreasing. **[4 marks; bookwork]**

**(a)(ii)** Now I estimate $T_n$ from below:

$$T_n = \sum_{k=1}^{n-1} \int_k^{k+1} f(k)\,dx + f(n) - \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,dx = \sum_{k=1}^{n-1} \int_k^{k+1} (f(k)-f(x))\,dx + f(n) \geq f(n).$$

**[8 marks; bookwork]**

**(a)(iii)** The above estimate and the fact that $f$ is nonnegative imply

$$T_n \geq 0.$$

**[2 marks; bookwork]**

**(a)(iv)** We are looking at a decreasing sequence $\{T_n\}$ which is bounded from below by zero. Such a sequence has to converge. **[1 marks; bookwork]**

**(a)(v)** Suppose that the series $\sum\limits_{k=1}^{\infty} f(k)$ converges. Then formula (4), the result from (a)(iv) and the algebra of limits tell us that $\lim_{n\to\infty} \int\limits_1^n f(x)\,dx$ exists.

Conversely, suppose that $\lim_{n\to\infty} \int\limits_1^n f(x)\,dx$ exists. Then formula (4), the result from (a)(iv) and the algebra of limits tell us that the series $\sum\limits_{k=1}^{\infty} f(k)$ converges. **[3 marks; bookwork]**

**(b)** For $\alpha \leq 1$ we have

$$\frac{1}{k^\alpha} \geq \frac{1}{k}, \qquad \forall k \geq 1,$$

so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ diverges by comparison with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

For $\alpha > 1$ we apply the Integral Test for the Convergence of Series (result from (a)(v)): $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ converges iff $\displaystyle\lim_{n\to\infty} \int_1^n \frac{dx}{x^\alpha}$ exists. We can apply the Integral Test for the Convergence of Series because for $\alpha > 1$ the function $f : [1, +\infty) \to [0, +\infty)$, $f(x) := \dfrac{1}{x^\alpha}$ is decreasing and takes nonnegative values.

We have

$$\int_1^n \frac{dx}{x^\alpha} = \frac{[x^{1-\alpha}]_1^n}{1-\alpha} = \frac{n^{1-\alpha}-1}{1-\alpha} \to \frac{1}{\alpha-1}$$

as $n \to \infty$.

The final answer is: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ converges iff $\alpha > 1$. **[7 marks; unseen exercise]**

## Solution to question 4

**(a)(i)** Consider two power series:

$$\sum_{n=0}^{\infty} a_n x^n \tag{5}$$

and

$$\sum_{n=1}^{\infty} n a_n x^n. \tag{6}$$

**Lemma 1** *The two power series (5) and (6) have the same radii of convergence.*

**Proof** Denote the radii of convergence of the power series (5) and (6) by $R$ and $R'$ respectively.

Let us prove first that

$$R' \leq R. \tag{7}$$

Suppose (7) is false. Then $R' > R$ and we can choose an $x$ such that $R < x < R'$. For this $x$ the series (6) converges absolutely whereas the series (5) diverges. But

$$|a_n x^n| \leq |n a_n x^n|,$$

$\forall n \in \mathbb{N}$, so the series (5) converges by the comparison test. Thus, the series (5) diverges and converges, which is impossible. This contradiction proves (7).

Let us prove now that

$$R \leq R'. \tag{8}$$

Suppose (8) is false. Then $R > R'$ and we can choose $y$, $z$ such that $R' < y < z < R$. For these $y$ and $z$ the series

$$\sum_{n=0}^{\infty} a_n z^n \tag{9}$$

converges absolutely whereas the series

$$\sum_{n=1}^{\infty} n a_n y^n \tag{10}$$

diverges. But

$$|na_n y^n| = |a_n z^n| \left( n \left| \frac{y}{z} \right|^n \right). \tag{11}$$

It is known from MATH1101 that $\lim\limits_{n\to\infty} \left( n \left| \frac{y}{z} \right|^n \right) = 0$ so there exists an $M \geq 0$ such that $n \left| \frac{y}{z} \right|^n \leq M$, $\forall n \in \mathbb{N}$, and formula (11) implies $|na_n y^n| \leq M|a_n z^n|$, $\forall n \in \mathbb{N}$. Hence the series (10) converges by the comparison test. Thus, the series (10) diverges and converges, which is impossible. This contradiction proves (8).

Formulae (7) and (8) imply $R = R'$. $\square$ **[9 marks; bookwork]**

**(a)(ii)** Now consider the power series

$$\sum_{n=1}^{\infty} na_n x^{n-1}. \tag{12}$$

Note that (12) is the formal derivative of the original power series (5).

**Lemma 2** *The two power series (5) and (12) have the same radii of convergence.*

**Proof** The two power series obviously converge for $x = 0$. For $x \neq 0$ we have

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} na_n x^n,$$

so the series in the left- and right-hand sides converge and diverge for the same $x$. Thus, the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} na_n x^{n-1}$ is the same as the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} na_n x^n$. It remains to note that, according to Lemma 1, the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} na_n x^n$ is the same as that of the original power series $\sum\limits_{n=0}^{\infty} a_n x^n$. $\square$ **[2 marks; bookwork]**

**(a)(iii)** Now consider the power series

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \tag{13}$$

where $k$ is an arbitrary fixed nonnegative integer number. Note that (12) is the formal $k$th derivative of the original power series (5).

**Lemma 3** *The two power series (5) and (13) have the same radii of convergence.*

**Proof** Repeated application of Lemma 2. $\square$ **[1 mark; bookwork]**

**(a)(iv)**

**Theorem 2** *Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then $f$ is differentiable on the interval $(-R, R)$ and $f'(x) = \sum\limits_{n=1}^{\infty} na_n x^{n-1}$.*

**Proof** Fix an $x_0 \in (-R, R)$ and an $h_0$ such that $0 < h_0 < R - |x_0|$ and let $h$ be such that $0 < |h| < h_0$. Then by the results from (a)(ii) and (a)(iii) and the inequality stated in the examination paper

$$\left| \frac{\sum\limits_{n=0}^{\infty} a_n (x_0+h)^n - \sum\limits_{n=0}^{\infty} a_n x_0^n}{h} - \sum_{n=1}^{\infty} n a_n x_0^{n-1} \right| = \left| \sum_{n=2}^{\infty} a_n \left( \frac{(x_0+h)^n - x_0^n}{h} - n x_0^{n-1} \right) \right|$$

$$\leq \sum_{n=2}^{\infty} \left| a_n \left( \frac{(x_0+h)^n - x_0^n}{h} - n x_0^{n-1} \right) \right| \leq \frac{|h|}{2} \sum_{n=2}^{\infty} n(n-1)|a_n|(|x_0| + |h|)^{n-2}$$

$$\leq \frac{|h|}{2} \underbrace{\sum_{n=2}^{\infty} n(n-1)|a_n|(|x_0| + h_0)^{n-2}}_{\text{number, not a function of } h} \to 0$$

as $h \to 0$, giving the required result. Note that the results from (a)(ii) and (a)(iii) play an important role in this proof: they guarantee that all the series appearing in the above argument converge absolutely. □ **[6 marks; bookwork]**

**(b)**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n .$$

**[3 marks; example done in lectures, though this is really known from school]**

**(c)** By (b), (a)(ii) and (a)(iv) we have

$$\left( \frac{1}{1-x} \right)' = \sum_{n=1}^{\infty} n x^{n-1} .$$

But

$$\left( \frac{1}{1-x} \right)' = \frac{1}{(1-x)^2} ,$$

so

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} .$$

**[4 marks; unseen exercise]**

**Solution to question 5**

**(a)**

**Theorem 3** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists a point $\xi \in (a, b)$ such that*

$$(g(b) - g(a)) f'(\xi) = (f(b) - f(a)) g'(\xi) . \tag{14}$$

**[3 marks; bookwork]**

**(b)**

**Theorem 4** *Let* $f, g: (a, b) \to \mathbb{R}$ *be differentiable and let* $c \in (a, b)$ *be such that* $f(c) = g(c) = 0$ *and* $g'(x) \neq 0$ *for* $x \neq c$. *Then*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{15}$$

*provided the latter limit exists.*

**[4 marks; bookwork]**

Explanation why $g(x) \neq 0$ for $x \neq c$ (this is not part of the proof of the theorem). Indeed, suppose $g(x) = 0$ for some $x \in (a, b)$, $x \neq c$.

*Case 1:* $x > c$. Apply Rolle's Theorem to the function $g$ on the interval $[c, x]$: Rolle's Theorem tells us that there exists a $\xi \in (c, x)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 4.

*Case 2:* $x < c$. Apply Rolle's Theorem to the function $g$ on the interval $[x, c]$: Rolle's Theorem tells us that there exists a $\xi \in (x, c)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 4.

**Proof of Theorem 4** Suppose $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists and equals $L$. Then

$$\lim_{x \to c^+} \frac{f'(x)}{g'(x)} = L \tag{16}$$

and

$$\lim_{x \to c^-} \frac{f'(x)}{g'(x)} = L. \tag{17}$$

In order to prove (15) we need to show that

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = L \tag{18}$$

and

$$\lim_{x \to c^-} \frac{f(x)}{g(x)} = L. \tag{19}$$

Let us prove (18).

Take an arbitrary sequence $\{x_n\} \subset (c, b)$ such that

$$\lim_{n \to \infty} x_n = c. \tag{20}$$

Applying Theorem 3 on the interval $[c, x_n]$ we get a sequence $\{\xi_n\}$ such that

$$c < \xi_n < x_n \tag{21}$$

and

$$(g(x_n) - g(c))f'(\xi_n) = (f(x_n) - f(c))g'(\xi_n)$$

which can be equivalently rewritten as

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(\xi_n)}{g'(\xi_n)}. \tag{22}$$

Formulae (20), (21) and the Sandwich Principle imply

$$\lim_{n \to \infty} \xi_n = c. \tag{23}$$

Formulae (16), (23) and the sequential definition of a limit (one-sided version) imply

$$\lim_{n\to\infty}\frac{f'(\xi_n)}{g'(\xi_n)} = L. \tag{24}$$

Formulae (22) and (24) imply

$$\lim_{n\to\infty}\frac{f(x_n)}{g(x_n)} = L. \tag{25}$$

Formula (25) and the sequential definition of a limit (one-sided version) imply[1] (18).

Formula (19) is proved similarly. □ **[9 marks; bookwork]**

**(c)(i)** Applying L'Hôpital's Rule once we get

$$\lim_{x\to 0}\frac{\sinh x}{x} = \lim_{x\to 0}\cosh x = \cosh 0 = 1.$$

**[2 marks; unseen exercise]**

**(c)(ii)** Applying L'Hôpital's Rule once and using the result from (c)(i) we get

$$\lim_{x\to 0}\frac{1-\cosh x}{x^2} = -\frac{1}{2}\lim_{x\to 0}\frac{\sinh x}{x} = -\frac{1}{2}\,.$$

**[2 marks; unseen exercise]**

**(c)(iii)** Using the result from (c)(i) we get

$$\begin{aligned}\lim_{x\to 0}\frac{\sinh^2 x-\tanh^2 x}{x^4} &= \lim_{x\to 0}\frac{\sinh^2(\cosh^2 x-1)}{x^4\cosh^2 x} = \lim_{x\to 0}\frac{\sinh^4 x}{x^4\cosh^2 x}\\ &= \left(\lim_{x\to 0}\frac{1}{\cosh x}\right)^2\left(\lim_{x\to 0}\frac{\sinh x}{x}\right)^4 = \left(\frac{1}{\cosh 0}\right)^2(1)^4 = 1.\end{aligned}$$

**[5 marks; unseen exercise]**

## Solution to question 6

**(a)**

**Definition 8** *A function* $f:[0,+\infty)\to\mathbb{R}$ *is said to be* locally Riemann integrable *over* $[0,+\infty)$ *if* $\forall a,b\in[0,+\infty)$ *we have* $f\in\mathcal{R}[a,b]$. *The set of all locally Riemann integrable functions over the interval* $[0,+\infty)$ *will be denoted by* $\mathcal{R}_{loc}[0,+\infty)$.

**[2 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval $I$ and not specifically for the interval $[0,+\infty)$]**

**(b)** We say that $f$ is integrable on $[0,+\infty)$ in the improper sense if $\lim\limits_{b\to+\infty}\int_0^b f(x)\,dx$ exists. We will denote this limit by $\displaystyle\int_0^{+\infty} f(x)\,dx$. **[4 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval $I$ and not specifically for the interval $[0,+\infty)$]**

[1] Here it is important that the sequence $\{x_n\}\subset(c,b)$ is an *arbitrary* sequence satisfying (20).

**(c)** Note that the integrand is continuous on $[0, +\infty)$, hence locally integrable on $[0, +\infty)$. So we have to check only integrability at $+\infty$. We have

$$\int_0^b \frac{1}{1+x^2}\,dx = \arctan b \to \frac{\pi}{2} \qquad \text{as} \qquad b \to +\infty,$$

so $\displaystyle\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$. **[2 marks; unseen exercise]**

**(d)**

**Theorem 5** *Let $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d \rangle$, and suppose that*

- $0 \le f(x) \le g(x)$ *for all $x \in I$;*
- $\int_c^d g(x)\,dx$ *exists.*

*Then $\int_c^d f(x)\,dx$ exists.*

**[3 marks; bookwork]**

**(e)**

**Theorem 6** *Suppose that $f \in \mathcal{R}_{loc}(I)$ and $|f|$ is integrable over $I$. Then $f$ is integrable over $I$.*

**[3 marks; bookwork]**

**(f)** Let us prove the existence of the improper integral

$$\int_0^{+\infty} \frac{\sin x}{1+x^2}\,dx\,. \tag{26}$$

Note that the integrand is continuous on $[0, +\infty)$, hence locally integrable on $[0, +\infty)$. So we have to check only integrability at $+\infty$.

By the result from (e), in order to prove the existence of the integral (26) it is sufficient to prove the existence of the integral

$$\int_0^{+\infty} \frac{|\sin x|}{1+x^2}\,dx\,. \tag{27}$$

Compare now the functions $\dfrac{|\sin x|}{1+x^2}$ and $\dfrac{1}{1+x^2}$ on the interval $[0, +\infty)$. We have

$$0 \le \frac{|\sin x|}{1+x^2} \le \frac{1}{1+x^2}\,,$$

so, by Theorem 5, in order to prove the existence of the integral (27) it is sufficient to prove the existence of the integral $\displaystyle\int_0^{+\infty} \frac{1}{1+x^2}\,dx$. But this has already been done in (c). **[6 marks; unseen exercise]**

**(g)** Note that the integrand is continuous on $[0, +\infty)$, hence locally integrable on $[0, +\infty)$. So we have to check only integrability at $+\infty$.

Integrating by parts we get

$$\int_0^b \left(\frac{\pi}{2} - \arctan x\right) \cos x\,dx = \left(\frac{\pi}{2} - \arctan b\right) \sin b + \int_0^b \frac{\sin x}{1+x^2}\,dx\,.$$

We have $\left(\dfrac{\pi}{2} - \arctan b\right) \sin b \to 0$ as $b \to +\infty$, so in order to prove the existence of the original integral we have to prove the existence of the integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{1+x^2}\,dx$. But this has already been done in (f). In fact, the numerical value of the improper integrals in (f) and (g) is the same. **[5 marks; unseen exercise]**