All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Explain, in terms of ε and δ, the statement "the function $f : [a, b] \to \mathbb{R}$ is continuous at the point $x_0 \in [a, b]$".
   (b) Explain the statement "the function $f : [a, b] \to \mathbb{R}$ is continuous".
   (c) Explain, in terms of ε and δ, the statement "the function $f : [a, b] \to \mathbb{R}$ is uniformly continuous".
   (d) Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that f is uniformly continuous. Here you may use without proof the Bolzano–Weierstrass Theorem.

2. (a) What is a partition of an interval $[a, b]$ and what is its mesh?
   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded.
      (i) Define the lower and upper Darboux sums of f with respect to a given partition of the interval $[a, b]$
      (ii) Define the lower and upper Riemann integrals of f over $[a, b]$.
      (iii) Define what it means for f to be Riemann integrable on $[a, b]$.
      (iv) State Riemann's Criterion for Integrability.
   (c) Define what it means for a function $f : [a, b] \to \mathbb{R}$ to be decreasing.
   (d) Prove that a decreasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. Here you may use without proof Riemann's Criterion for Integrability.
   (e) Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e & \text{if} \quad x = 0, \\ \sum_{n=0}^{[x^{-1}]} \frac{1}{n!} & \text{if} \quad x > 0, \end{cases}$$

   where $[x^{-1}]$ is the integer part of x^{-1}. Prove that this function is Riemann integrable on $[0, 1]$.

3. (a) Let $f : [1, +\infty) \to [0, +\infty)$ be a decreasing function and let

$$T_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)\, dx\,, \qquad n \in \mathbb{N}.$$

(i) Prove that the sequence $\{T_n\}$ is decreasing.

(ii) Prove that $T_n \geq f(n)$, $n \in \mathbb{N}$.

(iii) Prove that $T_n \geq 0$, $n \in \mathbb{N}$.

(iv) Prove that the sequence $\{T_n\}$ converges.

(v) Prove that the series $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\lim_{n\to\infty} \int_1^n f(x)\, dx$ exists.

(b) For what values of the parameter $\alpha \in \mathbb{R}$ does the series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ converge? Justify your answer.

4. (a) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.

(i) Prove that the power series $\sum_{n=1}^{\infty} n a_n x^n$ has radius of convergence R.

(ii) Prove that the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R.

(iii) Prove that the power series $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ has radius of convergence R.

(iv) Prove that the function $f : (-R, R) \to \mathbb{R}$ given by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is differentiable and that $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. You may use without proof the fact that

$$\left| \frac{(x_0 + h)^n - x_0^n}{h} - n x_0^{n-1} \right| \leq \frac{n(n-1)|h|}{2} (|x_0| + |h|)^{n-2},$$

$\forall x_0, h \in \mathbb{R}$, $h \neq 0$, $\forall n \in \mathbb{N}$, $n \geq 2$.

(b) Write down the power series for the function $f : (-1, 1) \to \mathbb{R}$ given by

$$f(x) = \frac{1}{1-x}\,.$$

(c) Write down the power series for the function $g : (-1, 1) \to \mathbb{R}$ given by

$$g(x) = \frac{1}{(1-x)^2}\,.$$

[*Hint:* you may find it helpful to use the results from (b) and (a)(iv).]

5. (a) State Cauchy's Generalisation of the Mean Value Theorem.

   (b) State and prove l'Hôpital's Rule for finding $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ when $f(c) = g(c) = 0$.

   (c) Evaluate the following limits:

      (i) $\lim_{x \to 0} \dfrac{\sinh x}{x}$,

      (ii) $\lim_{x \to 0} \dfrac{1 - \cosh x}{x^2}$,

      (iii) $\lim_{x \to 0} \dfrac{\sinh^2 x - \tanh^2 x}{x^4}$.

6. (a) Define what it means for a function $f : [0, +\infty) \to \mathbb{R}$ to be locally Riemann integrable.

   (b) Let $f : [0, +\infty) \to \mathbb{R}$ be locally Riemann integrable. Define what it means for f to be integrable over $[0, +\infty)$ in the improper sense.

   (c) Use the definition from (b) to prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{1}{1+x^2}\, dx$ and to evaluate this integral.

   (d) State the Comparison Theorem for Improper Integrals.

   (e) State the theorem relating the integrability, in the improper sense, of the functions f and $|f|$.

   (f) Prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{1+x^2}\, dx$.

   [*Hint:* you may find it helpful to use the results from (c), (d) and (e).]

   (g) Prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \left(\frac{\pi}{2} - \arctan x\right) \cos x \, dx$.

   [*Hint:* you may find it helpful to integrate by parts.]