MATH1102 Analysis

Solutions to examination paper for the academic year 2012/2013

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Solution to question 1

(a)(i)

Definition 1 *A sequence of real numbers* $\{x_n\}$ *is said to be a* Cauchy sequence *if* $\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N}$ *such that*

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < \varepsilon.$$

[4 marks; bookwork]

(a)(ii)

Theorem 1 *A sequence of real numbers* $\{x_n\}$ *converges iff it is a Cauchy sequence.*

[4 marks; bookwork]

(b)

Definition 2 *We say that the function* $f : (-\infty, 0) \to \mathbb{R}$ *is* uniformly continuous *if* $\forall \varepsilon > 0 \ \ \exists \delta > 0$ *such that*

$$|x - y| < \delta \quad \& \quad x, y \in (-\infty, 0) \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon.$$

[4 marks; bookwork]

(c) Let ε be an arbitrary positive number. The definition of uniform continuity (Definition 2) tells us that there exists a $\delta > 0$ such that

$$|x - y| < \delta \quad \& \quad x, y \in (-\infty, 0) \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon. \tag{1}$$

The sequence $\{x_n\}$ is convergent, hence, by Cauchy's General Principle of Convergence (Theorem 1) it is a Cauchy sequence. The definition of a Cauchy sequence (Definition 1) tells us that there exists an $N \in \mathbb{N}$ such that

$$m, n > N \qquad \Longrightarrow \qquad |x_n - x_m| < \delta, \tag{2}$$

where the δ is the one from formula (1). Combining formulae (1) and (2) we get

$$m, n > N \qquad \Longrightarrow \qquad |f(x_n) - f(x_m)| < \varepsilon,$$

which in view of Definition 1, means that $\{f(x_n)\}$ is a Cauchy sequence. Cauchy's General Principle of Convergence (Theorem 1) now tells us that the sequence $\{f(x_n)\}$ converges. **[13 marks; similar to an exercise from an exercise sheet]**

Solution to question 2

(a) The *lower Darboux sum* of f with respect to the partition $P = \{x_0, x_1, \dots, x_n\}$ is defined as

$$L(f,P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \, ,$$

where $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$. The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f,P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \, ,$$

where $M_i := \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$. **[4 marks; bookwork]**

(b) Write the original partition P as

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the partition P' as

$$a = x_0 < x_1 < \cdots < x_{k-1} < x' < x_k < \cdots < x_{n-1} < x_n = b$$

Then

$$\begin{aligned} L(f,P') - L(f,P) = \left(\inf_{x \in [x_{k-1}, x']} f(x) \right) (x' - x_{k-1}) + \left(\inf_{x \in [x', x_k]} f(x) \right) (x_k - x') \\ - \left(\inf_{x \in [x_{k-1}, x_k]} f(x) \right) (x_k - x_{k-1}). \end{aligned}$$

But

$$\inf_{x \in [x_{k-1}, x']} f(x) \geq \inf_{x \in [x_{k-1}, x_k]} f(x), \qquad \inf_{x \in [x', x_k]} f(x) \geq \inf_{x \in [x_{k-1}, x_k]} f(x),$$

so

$$\begin{aligned} L(f,P') - L(f,P) \geq \left(\inf_{x \in [x_{k-1}, x_k]} f(x) \right) (x' - x_{k-1}) + \left(\inf_{x \in [x_{k-1}, x_k]} f(x) \right) (x_k - x') \\ - \left(\inf_{x \in [x_{k-1}, x_k]} f(x) \right) (x_k - x_{k-1}) = 0. \end{aligned}$$

The inequality $U(f,P') \leq U(f,P)$ is proved in a similar way. **[7 marks; bookwork]**

(c) Repeated application of the result from (b). **[1 mark; bookwork]**

(d) Let $R := P \cup Q$ be a common refinement of P and Q. Then, by the result from (c),

$$L(f,P) \leq L(f,R), \qquad U(f,R) \leq U(f,Q).$$

But, obviously, $L(f,R) \leq U(f,R)$, so

$$L(f,P) \leq L(f,R) \leq U(f,R) \leq U(f,Q)$$

which implies $L(f,P) \leq U(f,Q)$. **[4 marks; bookwork]**

(e)

Definition 3 *Let* $f \in \mathcal{B}[a,b]$*. The* lower Riemann integral *of* f *over* $[a,b]$ *is defined as*

$$\underline{\int_a^b} f(x)\,dx := \sup_{P \in \mathcal{P}[a,b]} L(f,P)\,.$$

The upper Riemann integral *of* f *over* $[a,b]$ *is defined as*

$$\overline{\int_a^b} f(x)\,dx := \inf_{P \in \mathcal{P}[a,b]} U(f,P)\,.$$

[3 marks; bookwork]

(f) By the result from (d), $L(f,P) \le U(f,Q)$ for any pair of partitions P and Q. Taking the supremum over $P \in \mathcal{P}[a,b]$ we get

$$\underline{\int_a^b} f(x)\,dx \le U(f,Q). \tag{3}$$

Note that formula (3) holds for any partition Q. Taking the infimum over $Q \in \mathcal{P}[a,b]$ we get

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx\,.$$

[4 marks; bookwork]

(g)

Definition 4 *The function* $f \in \mathcal{B}[a,b]$ *is said to be* Riemann integrable *on* $[a,b]$ *if*

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx\,.$$

In this case the common value of the lower and upper Riemann integrals is denoted by

$$\int_a^b f(x)\,dx$$

and is called the Riemann integral *of* f *over* $[a,b]$*. The set of all Riemann integrable functions* $f : [a,b] \to \mathbb{R}$ *will be denoted by* $\mathcal{R}[a,b]$*.*

[2 marks; bookwork]

Solution to question 3

(a)

Definition 5 *Let* $f : I \to \mathbb{R}$*. A* primitive *of* f *is a function* $F : I \to \mathbb{R}$ *which is continuous on* I*, differentiable on the interior of* I *and satisfies* $F'(x) = f(x)$ *on the interior of* I*.*

[3 marks; bookwork]

(b) No: one can always add a constant to F. **[1 mark; bookwork]**

(c)

Theorem 2 *Let $f \in \mathcal{R}[a,b]$ and let F be a primitive of f. Then*

$$\int_a^b f(x)\,dx \;=\; F(b)-F(a) = F(x)|_a^b\,.$$

[3 marks; bookwork]

(d)

Definition 6 *Let $f \in \mathcal{R}_{loc}(I)$. An* indefinite integral *of f is a function $F : I \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\,dt$$

for some $a \in I$.

[2 marks; bookwork]

(e) As the interval I is open, it suffices to prove that an indefinite integral

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable and satisfies $F'(x) = f(x)$, $\forall x \in I$. There is no need for a separate proof of continuity of F as it follows from differentiability.

Let us fix an arbitrary $x_0 \in I$. Let ε be an arbitrary positive number. Then $\exists \delta > 0$ such that $(x_0-\delta, x_0+\delta) \subset I$ and

$$|x-x_0| < \delta \qquad \Longrightarrow \qquad |f(x)-f(x_0)| < \frac{\varepsilon}{2}$$

(here I used the fact that f is continuous at x_0). Hence, for any $x \in (x_0-\delta, x_0+\delta)$, $x \neq x_0$ we have

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| = \left|\frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x-x_0} - \frac{\int_{x_0}^x f(x_0)\,dt}{x-x_0}\right| = \left|\frac{\int_{x_0}^x (f(t)-f(x_0))\,dt}{x-x_0}\right|$$
$$\le \frac{\left|\int_{x_0}^x |f(t)-f(x_0)|\,dt\right|}{|x-x_0|} \le \frac{\left|\int_{x_0}^x \frac{\varepsilon}{2}\,dt\right|}{|x-x_0|} = \frac{\frac{\varepsilon}{2}|x-x_0|}{|x-x_0|} = \frac{\varepsilon}{2} < \varepsilon.$$

By the definition of a limit this means that

$$\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0),$$

which, in turn, by the definition of a derivative, means that $F'(x_0) = f(x_0)$. **[11 marks; version of a theorem proved in the lectures; this particular version unseen]**

(f) Put

$$F(y) := \int_0^y \sin(t^3)\,dt\,.$$

Then we have

$$G(x) = F(\sin(x^2)) - F(\sin x).$$

Hence, by the Chain Rule and part (e) we have

$$G'(x) = 2x(\cos x^2)F'(\sin(x^2)) - (\cos x)F'(\sin x) = 2x(\cos x^2)\sin(\sin^3(x^2)) - (\cos x)\sin(\sin^3 x).$$

[5 marks; unseen exercise]

Solution to question 4

(a) Put $h(x) := f(x) - g(x)$. Then $h'(x) = 0$ for all $x \in (-1, 1)$. I claim that

$$h(x) = h(0), \qquad \forall x \in (-1, 1). \tag{4}$$

Case 1: $x = 0$. Obviously, (4) is true.

Case 2: $x \in (0, 1)$. Applying the Mean Value Theorem to the function h on the interval $[0, x]$ we get

$$\frac{h(x) - h(0)}{x - 0} = h'(\xi) \tag{5}$$

for some $\xi \in (0, x)$. But $h'(\xi) = 0$, so (5) implies (4).

Case 3: $x \in (-1, 0)$. Applying the Mean Value Theorem to the function h on the interval $[x, 0]$ we get

$$\frac{h(0) - h(x)}{0 - x} = h'(\xi) \tag{6}$$

for some $\xi \in (x, 0)$. But $h'(\xi) = 0$, so (6) implies (4). **[8 marks; unseen exercise]**

(b) In the lectures the radius of convergence was defined as follows. Introduce the set

$$S := \left\{ r \geq 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ converges for } x = r \text{ or } x = -r \right\}. \tag{7}$$

Obviously, $\{0\} \subset S \subset [0, +\infty)$. The *radius of convergence* of a power series is the extended real number

$$R := \sup S$$

where S is the set (7).

Another possible definition is that the radius of convergence of a power series is the extended real number R such that for $|x| < R$ the series converges absolutely and for $|x| > R$ the series diverges. I will accept this definition even though it is not *a priori* obvious that an R with these properties exists. **[3 marks; bookwork]**

(c) Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$. **[4 marks; bookwork]**

(d) Fix an $x \neq 0$ and apply the ratio test:

$$\left| \frac{\frac{(-1)^{k+2}}{k+1} x^{k+1}}{\frac{(-1)^{k+1}}{k} x^k} \right| = \frac{k}{k+1} |x| = \frac{|x|}{1 + \frac{1}{k}} \to |x|$$

as $k \to \infty$. Hence, the power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is 1. **[4 marks; unseen exercise]**

(e) The results from (c) and (d) tell us that the function g is differentiable on $(-1, 1)$ and that its derivative is obtained by differentiating the power series for g term-by-term:

$$g'(x) = \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k \right)' = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} \right)' = \sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x}$$

because we are looking at a geometric series. **[4 marks; unseen exercise]**

(f) Application of the result from (a) with $f(x) = \ln(1+x)$ and $g(x)$ as in (e) gives

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, x^k \, + c\,, \qquad \forall x \in (-1,1).$$

Setting $x = 0$ we conclude that $c = 0$. **[2 marks; unseen exercise]**

Solution to question 5

(a) Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that

$$(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)\,.$$

[4 marks; bookwork]

(b) Fix $x > 0$ and consider the function

$$\begin{aligned} g(t) := f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - \dots \\ - \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x-t)^n\,, \quad (8) \end{aligned}$$

where $t \in (a,b)$ is the independent variable. Since $f : (a,b) \to \mathbb{R}$ is $n+1$ times differentiable, $g : (a,b) \to \mathbb{R}$ is differentiable. Differentiating (8) we get

$$\begin{aligned} g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \dots \\ \dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] \\ = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n \end{aligned}$$

(all the terms cancelled out, apart from the last one). Applying Cauchy's Generalisation of the Mean Value Theorem to the functions $g(t)$ and $h(t) := (x-t)^{n+1}$ on the interval $[0,x]$ (which we can because g is continuous on $[0,x] \subset (a,b)$ and differentiable on $(0,x) \subset (a,b)$) we get

$$\frac{g(x) - g(0)}{h(x) - h(0)} = \frac{g'(\xi)}{h'(\xi)} = \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{-(n+1)(x-\xi)^n} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi \in (0,x)$. But

$$g(0) = R_n(x), \quad g(x) = 0, \quad h(0) = x^{n+1}, \quad h(x) = 0,$$

so the above formula can be rewritten as

$$\frac{-R_n(x)}{-x^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,,$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\, x^{n+1}$. **[11 marks; bookwork]**

(c) We have

$$\arctan' x = \frac{1}{1+x^2}\,, \qquad \arctan'' x = -\frac{2x}{(1+x^2)^2}\,, \qquad \arctan''' x = \frac{2(3x^2-1)}{(1+x^2)^3}\,,$$

so application of the result from (b) to the function $\arctan : \mathbb{R} \to \mathbb{R}$ with $n = 2$ gives

$$\arctan x = x + \frac{3\xi^2 - 1}{3(1+\xi^2)^3}\, x^3 \tag{9}$$

for some $\xi \in (0, x)$. **[5 marks; unseen exercise]**

(d) Take $x = \frac{1}{\sqrt{3}}$. Then formula (9) becomes

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1-3\xi^2}{9\sqrt{3}(1+\xi^2)^3}$$

for some $\xi \in (0, \frac{1}{\sqrt{3}})$. The function $\xi \mapsto \frac{1-3\xi^2}{(1+\xi^2)^3}$ is strictly decreasing on the interval $[0, \frac{1}{\sqrt{3}}]$ (the numerator decreases as ξ increases, whereas the denominator increases as ξ increases), so

$$\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}\,,$$

or, equivalently, $\dfrac{16}{9}\sqrt{3} < \pi < 2\sqrt{3}$. **[5 marks; unseen exercise]**

Solution to question 6

(a)

Definition 7 *A function $f : I \to \mathbb{R}$ is said to be* locally Riemann integrable *over I if $\forall a, b \in I$ we have $f \in \mathcal{R}[a,b]$. The set of all locally Riemann integrable functions over the interval I will be denoted by $\mathcal{R}_{loc}(I)$.*

[2 marks; bookwork]

(b)

Definition 8 *Let $f \in \mathcal{R}_{loc}(I)$, $I = \langle c, d \rangle$.*

(i) We say that f is integrable at c in the improper sense if $\exists p \in I$, $c < p$, such that $\lim_{a \to c^+} \int_a^p f(x)\,dx$ exists. We will denote this limit by $\int_c^p f(x)\,dx$.

(ii) We say that f is integrable at d in the improper sense if $\exists p \in I$, $p < d$, such that $\lim_{b \to d^-} \int_p^b f(x)\,dx$ exists. We will denote this limit by $\int_p^d f(x)\,dx$.

(iii) We say that f is integrable over I in the improper sense if both (i) and (ii) hold for the same p. We will then write $\int_c^d f(x)\,dx := \int_c^p f(x)\,dx + \int_p^d f(x)\,dx$.

[5 marks; bookwork]

(c)

Theorem 3 *Let $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d\rangle$, and suppose that*

- $0 \le f(x) \le g(x)$ *for all* $x \in I$;
- $\int_c^d g(x)\,dx$ *exists.*

Then $\int_c^d f(x)\,dx$ *exists.*

[3 marks; bookwork]

Proof Given arbitrary $a, b \in I$, $a < b$, we have

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx\,. \tag{10}$$

But according to the fact stated in the examination paper there exists a K such that

$$\int_a^b g(x)\,dx \le K \tag{11}$$

irrespective of the choice of a, b. Combining formulae (10) and (11) we get

$$\int_a^b f(x)\,dx \le K$$

irrespective of the choice of a, b. But according to the fact stated in the examination paper this implies that $\int_c^d f(x)\,dx$ exists. $\square$ **[4 marks; bookwork]**

(d) Introduce the functions

$$f^+(x) := \frac{|f(x)| + f(x)}{2}\,, \qquad f^-(x) := \frac{|f(x)| - f(x)}{2}\,.$$

It is easy to see that

$$0 \le f^+(x) \le |f(x)|, \tag{12}$$

$$0 \le f^-(x) \le |f(x)| \tag{13}$$

(to check the above inequalities consider separately the cases $f(x) \ge 0$ and $f(x) < 0$). By the Theorem about the Properties of the Riemann Integral we have $f^+, f^- \in \mathcal{R}_{loc}(I)$. Formula (12) and the Comparison Theorem for Improper Integrals (Theorem 3) imply that f^+ is integrable over I. Similarly, formula (13) implies that f^- is integrable over I. But $f(x) = f^+(x) - f^-(x)$, so the integrability of f over I follows by linearity. $\square$ **[5 marks; bookwork]**

(e) Let us prove the existence of the improper integral

$$\int_0^{+\infty} \frac{\sin x}{x + x^2}\,dx\,. \tag{14}$$

We have

$$\lim_{x\to 0^+} \frac{\sin x}{x + x^2} = \lim_{x\to 0^+} \frac{\sin x}{x} \lim_{x\to 0^+} \frac{1}{1+x} = 1,$$

so there is no problem at $x = 0$. Hence, in order to prove the existence of the integral (14) it is sufficient to prove the existence of the integral

$$\int_1^{+\infty} \frac{\sin x}{x + x^2}\,dx\,. \tag{15}$$

By the result from (d), in order to prove the existence of the integral (15) it is sufficient to prove the existence of the integral

$$\int_1^{\infty} \frac{|\sin x|}{x+x^2}\, dx\,. \tag{16}$$

Compare now the functions $\dfrac{|\sin x|}{x+x^2}$ and $\dfrac{1}{x^2}$ on the interval $[1,+\infty)$ We have

$$0 \le \frac{|\sin x|}{x+x^2} \le \frac{1}{x^2}$$

so, by Theorem 3, in order to prove the existence of the integral (16) it is sufficient to prove the existence of the integral

$$\int_1^{+\infty} \frac{1}{x^2}\, dx\,. \tag{17}$$

But the existence of the integral (17) is a standard fact. **[6 marks; unseen exercise]**