MATH1102 Analysis

Solutions to examination paper for the academic year 2011/2012

Dmitri Vassiliev

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Solution to question 1

(a)(i) We say that $L = \lim_{x \to x_0} f(x)$ if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon.$$

[3 marks; bookwork]

(a)(ii) We say that the function $f : D \to \mathbb{R}$ is *continuous* at the point x_0 if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

[3 marks; bookwork]

(a)(iii) The function f is said to be *differentiable* at the point x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In this case the limit is denoted by $f'(x_0)$ (or $\frac{df}{dx}(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$) and is called the *derivative* of f at x_0. **[3 marks; bookwork]**

(a)(iv) Let f be differentiable at the point x_0. Then, by the Algebra of Limits, we have

$$0 = f'(x_0) \cdot 0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) =$$
$$\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right) = \lim_{x \to x_0} (f(x) - f(x_0)),$$

so

$$\lim_{x \to x_0} (f(x) - f(x_0)) = 0.$$

But we also have the trivial identity

$$\lim_{x \to x_0} f(x_0) = f(x_0).$$

Adding up the latter two identities we get $\lim_{x \to x_0} f(x) = f(x_0)$, which means that the function is continuous at the point x_0. **[7 marks; bookwork]**

(b)(i) The ε, δ definition of a limit works here with $L = 0$ and $\delta = \varepsilon$. Hence, $\lim_{x\to 0} f(x) = 0$. **[3 marks; unseen exercise]**

(b)(ii) The function f is not continuous at the point 0 because $\lim_{x\to 0} f(x) = 0 \neq 1 = f(0)$. **[3 marks; unseen exercise]**

(b)(iii) The function f is not differentiable at the point 0 because it is not continuous at this point. **[3 marks; unseen exercise]**

Solution to question 2

(a)(i) Consider two power series:

$$\sum_{n=0}^{\infty} a_n x^n \tag{1}$$

and

$$\sum_{n=1}^{\infty} n a_n x^n. \tag{2}$$

Lemma 1 *The two power series (1) and (2) have the same radii of convergence.*

Proof Denote the radii of convergence of the power series (1) and (2) by R and R' respectively. Let us prove first that

$$R' \leq R. \tag{3}$$

Suppose (3) is false. Then $R' > R$ and we can choose an x such that $R < x < R'$. For this x the series (2) converges absolutely whereas the series (1) diverges. But

$$|a_n x^n| \leq |n a_n x^n|,$$

$\forall n \in \mathbb{N}$, so the series (1) converges by the comparison test. Thus, the series (1) diverges and converges, which is impossible. This contradiction proves (3).

Let us prove now that

$$R \leq R'. \tag{4}$$

Suppose (4) is false. Then $R > R'$ and we can choose y, z such that $R' < y < z < R$. For these y and z the series

$$\sum_{n=0}^{\infty} a_n z^n \tag{5}$$

converges absolutely whereas the series

$$\sum_{n=1}^{\infty} n a_n y^n \tag{6}$$

diverges. But

$$|n a_n y^n| = |a_n z^n| \left(n \left| \frac{y}{z} \right|^n \right). \tag{7}$$

It is known from MATH1101 that $\lim_{n\to\infty} \left(n \left| \frac{y}{z} \right|^n \right) = 0$ so there exists an $M \geq 0$ such that $n \left| \frac{y}{z} \right|^n \leq M$, $\forall n \in \mathbb{N}$, and formula (7) implies $|n a_n y^n| \leq M |a_n z^n|$, $\forall n \in \mathbb{N}$. Hence the series (6) converges by the comparison test. Thus, the series (6) diverges and converges, which is impossible. This contradiction proves (4).

Formulae (3) and (4) imply $R = R'$. $\Box$ **[9 marks; bookwork]**

(a)(ii) Now consider the power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1}. \tag{8}$$

Note that (8) is the formal derivative of the original power series (1).

Lemma 2 *The two power series (1) and (8) have the same radii of convergence.*

Proof The two power series obviously converge for $x = 0$. For $x \neq 0$ we have

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} n a_n x^n,$$

so the series in the left- and right-hand sides converge and diverge for the same x. Thus, the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ is the same as the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} n a_n x^n$. It remains to note that, according to Lemma 1, the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} n a_n x^n$ is the same as that of the original power series $\sum\limits_{n=0}^{\infty} a_n x^n$. □ **[2 marks; bookwork]**

(a)(iii) Now consider the power series

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \tag{9}$$

where k is an arbitrary fixed nonnegative integer number. Note that (8) is the formal kth derivative of the original power series (1).

Lemma 3 *The two power series (1) and (9) have the same radii of convergence.*

Proof Repeated application of Lemma 2. □ **[1 mark; bookwork]**

(a)(iv)

Theorem 1 *Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$.*

Proof Fix an $x_0 \in (-R, R)$ and an h_0 such that $0 < h_0 < R - |x_0|$ and let h be such that $0 < |h| < h_0$. Then by the results from (a)(ii) and (a)(iii) and the inequality stated in the examination paper

$$\left| \frac{\sum\limits_{n=0}^{\infty} a_n (x_0 + h)^n - \sum\limits_{n=0}^{\infty} a_n x_0^n}{h} - \sum_{n=1}^{\infty} n a_n x_0^{n-1} \right| = \left| \sum_{n=2}^{\infty} a_n \left(\frac{(x_0 + h)^n - x_0^n}{h} - n x_0^{n-1} \right) \right|$$

$$\leq \sum_{n=2}^{\infty} \left| a_n \left(\frac{(x_0 + h)^n - x_0^n}{h} - n x_0^{n-1} \right) \right| \leq \frac{|h|}{2} \sum_{n=2}^{\infty} n(n-1)|a_n|(|x_0| + |h|)^{n-2}$$

$$\leq \frac{|h|}{2} \underbrace{\sum_{n=2}^{\infty} n(n-1)|a_n|(|x_0| + h_0)^{n-2}}_{\text{number, not a function of } h} \to 0$$

as $h \to 0$, giving the required result. Note that the results from (a)(ii) and (a)(iii) play an important role in this proof: they guarantee that all the series appearing in the above argument converge absolutely. $\Box$ **[6 marks; bookwork]**

(b) We know that

$$\cos(x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{(2k)!} \tag{10}$$

and that the radius of convergence of this power series is ∞ (this follows from Example 2.10 from the lecture notes).

Let us now try *guessing* a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = \cos(x^2)$ and $f(0) = 0$. Examination of (10) indicates that a good guess would be

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+1}}{(2k)!\,(4k+1)}\,. \tag{11}$$

We will now show that the function (11) is well defined for all $x \in \mathbb{R}$, that it is differentiable and that it satisfies the equation $f'(x) = \cos(x^2)$ on $\mathbb{R}$ as well as the condition $f(0) = 0$.

By (a)(ii) the power series (11) and (10) have the same radii of convergence, hence the radius of convergence of (11) is ∞.

The fact that the function (11) satisfies the condition $f(0) = 0$ is obvious as the power series in the RHS of (11) does not contain a term with x^0.

Differentiability of the function (11) follows from (a)(iv). Theorem 1 also tells us that we can differentiate the series (11) term-by-term within the interval of convergence (which in our case is the whole real line $\mathbb{R}$) so that

$$f'(x) = \sum_{k=0}^{\infty} \left((-1)^k \frac{x^{4k+1}}{(2k)!\,(4k+1)} \right)' = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{(2k)!} = \cos(x^2).$$

[7 marks; unseen exercise]

Solution to question 3

(a)

Theorem 2 *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $\xi \in (a, b)$ such that*

$$(g(b) - g(a))f'(\xi) = (f(b) - f(a))g'(\xi)\,. \tag{12}$$

[3 marks; bookwork]

(b)

Theorem 3 *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable and let $c \in (a, b)$ be such that $f(c) = g(c) = 0$ and $g'(x) \neq 0$ for $x \neq c$. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{13}$$

provided the latter limit exists.

[4 marks; bookwork]

Explanation why $g(x) \neq 0$ for $x \neq c$ (this is not part of the proof of the theorem). Indeed, suppose $g(x) = 0$ for some $x \in (a, b)$, $x \neq c$.

Case 1: $x > c$. Apply Rolle's Theorem to the function g on the interval $[c, x]$: Rolle's Theorem tells us that there exists a $\xi \in (c, x)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 3.

Case 2: $x < c$. Apply Rolle's Theorem to the function g on the interval $[x, c]$: Rolle's Theorem tells us that there exists a $\xi \in (x, c)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 3.

Proof of Theorem 3 Suppose $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists and equals L. Then

$$\lim_{x\to c^+} \frac{f'(x)}{g'(x)} = L \tag{14}$$

and

$$\lim_{x\to c^-} \frac{f'(x)}{g'(x)} = L. \tag{15}$$

In order to prove (13) we need to show that

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = L \tag{16}$$

and

$$\lim_{x\to c^-} \frac{f(x)}{g(x)} = L. \tag{17}$$

Let us prove (16).

Take an arbitrary sequence $\{x_n\} \subset (c, b)$ such that

$$\lim_{n\to\infty} x_n = c. \tag{18}$$

Applying Theorem 2 on the interval $[c, x_n]$ we get a sequence $\{\xi_n\}$ such that

$$c < \xi_n < x_n \tag{19}$$

and

$$(g(x_n) - g(c))f'(\xi_n) = (f(x_n) - f(c))g'(\xi_n)$$

which can be equivalently rewritten as

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(\xi_n)}{g'(\xi_n)}. \tag{20}$$

Formulae (18), (19) and the Sandwich Principle imply

$$\lim_{n\to\infty} \xi_n = c. \tag{21}$$

Formulae (14), (21) and the sequential definition of a limit (one-sided version) imply

$$\lim_{n\to\infty} \frac{f'(\xi_n)}{g'(\xi_n)} = L. \tag{22}$$

Formulae (20) and (22) imply

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L. \tag{23}$$

Formula (23) and the sequential definition of a limit (one-sided version) imply[1] (16).

Formula (17) is proved similarly. $\square$ **[9 marks; bookwork]**

[1] Here it is important that the sequence $\{x_n\} \subset (c, b)$ is an *arbitrary* sequence satisfying (18).

(c)(i) Applying L'Hôpital's Rule three times we get

$$\lim_{x\to 0}\frac{2x+x^2+2\ln(1-x)}{x^3}=\lim_{x\to 0}\frac{2+2x-\frac{2}{1-x}}{3x^2}=\lim_{x\to 0}\frac{2-\frac{2}{(1-x)^2}}{6x}=\lim_{x\to 0}\frac{-\frac{4}{(1-x)^3}}{6}=-\frac{2}{3}\,.$$

[3 marks; unseen exercise]

(c)(ii) We have

$$\frac{\tan^2 x-\arctan^2 x}{x^4}=\frac{\tan x-\arctan x}{x^3}\;\frac{\tan x+\arctan x}{x}\,. \tag{24}$$

Applying in each case L'Hôpital's Rule once we get

$$\lim_{x\to 0}\frac{\tan x+\arctan x}{x}=\lim_{x\to 0}\frac{\frac{1}{\cos^2 x}+\frac{1}{1+x^2}}{1}=2, \tag{25}$$

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x-\arctan x}{x^3}&=\lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-\frac{1}{1+x^2}}{3x^2}=\lim_{x\to 0}\frac{1}{(1+x^2)\cos^2 x}\;\lim_{x\to 0}\frac{1+x^2-\cos^2 x}{3x^2}\\&=\lim_{x\to 0}\frac{x^2+\sin^2 x}{3x^2}=\frac{1}{3}+\frac{1}{3}\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2=\frac{2}{3}\,.\end{aligned} \tag{26}$$

Note that in (26) we used the following standard fact (proved by applying L'Hôpital's Rule once):

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Combining formulae (24)–(26) we get

$$\lim_{x\to 0}\frac{\tan^2 x-\arctan^2 x}{x^4}=\frac{4}{3}\,.$$

[6 marks; unseen exercise]

Solution to question 4

(a)

Theorem 4 *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $\xi\in(a,b)$ such that*

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}\,. \tag{27}$$

[3 marks; bookwork]

(b) Fix $x\in(a,0)$ and consider the function

$$\begin{aligned}g(t):=f(x)-f(t)-f'(t)(x-t)-\frac{f''(t)}{2!}(x-t)^2-\frac{f'''(t)}{3!}(x-t)^3-\dots\\-\frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1}-\frac{f^{(n)}(t)}{n!}(x-t)^n\,,\end{aligned} \tag{28}$$

where t is the independent variable. Since $f : (a, b) \to \mathbb{R}$ is $n+1$ times differentiable, $g : (a, b) \to \mathbb{R}$ is differentiable. Differentiating (28) we get

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \dots$$
$$\dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right]$$
$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

(all the terms cancelled out, apart from the last one). Applying Theorem 4 to the function $g(t)$ on the interval $[x, 0]$ (which we can because g is differentiable on (a, b), hence continuous on $[x, 0] \subset (a, b)$ and differentiable on $(x, 0) \subset (a, b)$) we get

$$\frac{g(0) - g(x)}{0 - x} = g'(\xi) = -\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$$

for some $\xi \in (x, 0)$. But

$$g(0) = R_n(x), \quad g(x) = 0,$$

so the above formula can be rewritten as

$$\frac{R_n(x)}{-x} = -\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n\,,$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n x\,.$ □ **[12 marks; bookwork; note that in the lectures the above argument was done briefly (I did in detail Lagrange's form of the remainder) and only for the case $x > 0$]**

(c) Application of the result from (b) to the function $f : (-1, 99) \to \mathbb{R}$, $f(x) = \ln(1 + x)$ with $x = -\frac{3}{4}$ gives

$$\ln\left(\frac{1}{2}\right) = -\sum_{k=1}^{n} \frac{3^n}{n4^n} + R_n\left(-\frac{3}{4}\right),$$

$$R_n\left(-\frac{3}{4}\right) = (-1)^n \frac{\left(-\frac{3}{4} - \xi\right)^n \left(-\frac{3}{4}\right)}{(1+\xi)^{n+1}} = -\frac{\frac{3}{4}}{1+\xi}\left(\frac{\frac{3}{4}+\xi}{1+\xi}\right)^n$$

for some $\xi \in (-3/4, 0)$. We have

$$\max_{\xi \in [-\frac{3}{4}, 0]} \frac{\frac{3}{4}+\xi}{1+\xi} = \frac{3}{4}$$

(here the function being maximised has no critical points and achieves its maximum at the endpoint $\xi = 0$), so

$$\left|R_n\left(-\frac{3}{4}\right)\right| \le \frac{\left(\frac{3}{4}\right)^{n+1}}{1+\xi} \le 4\left(\frac{3}{4}\right)^{n+1} \to 0 \qquad \text{as} \qquad n \to \infty,$$

and, hence,

$$R_n\left(-\frac{3}{4}\right) \to 0 \qquad \text{as} \qquad n \to \infty$$

which gives the required result. **[10 marks; this question is a special case of a question from an exercise sheet]**

Solution to question 5

(a) A *partition* P of the interval $[a, b]$ is a finite sequence of real numbers $P = \{x_0, x_1, \dots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

[2 marks; bookwork]

(b)(i) The *lower Darboux sum* of f with respect to the partition P is defined as

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}) ,$$

where $m_i := \inf_{x \in [x_{i-1}, x_i]} f(x)$. The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}) ,$$

where $M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$. **[3 marks; bookwork]**

(b)(ii) The *lower Riemann integral* of f over $[a, b]$ is defined as

$$\underline{\int_a^b} f(x)\, dx := \sup_{P \in \mathcal{P}[a,b]} L(f, P) .$$

The *upper Riemann integral* of f over $[a, b]$ is defined as

$$\overline{\int_a^b} f(x)\, dx := \inf_{P \in \mathcal{P}[a,b]} U(f, P) .$$

[3 marks; bookwork]

(b)(iii) The function f is said to be *Riemann integrable* on $[a, b]$ if

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx .$$

In this case the common value of the lower and upper Riemann integrals is denoted by

$$\int_a^b f(x)\, dx$$

and is called the *Riemann integral* of f over $[a, b]$. The set of all Riemann integrable functions $f : [a, b] \to \mathbb{R}$ will be denoted by $\mathcal{R}[a, b]$. **[2 marks; bookwork]**

(b)(iv)

Theorem 5 *Let $f \in \mathcal{B}[a, b]$. Then $f \in \mathcal{R}[a, b]$ iff $\forall \varepsilon > 0\ \ \exists P \in \mathcal{P}[a, b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon. \tag{29}$$

[3 marks; bookwork]

(c) A function $f : [a, b] \to \mathbb{R}$ is said to be *decreasing* on $[a, b]$ if

$$x_1 < x_2 \quad \& \quad x_1, x_2 \in [a, b] \qquad \Longrightarrow \qquad f(x_1) \geq f(x_2).$$

[2 marks; bookwork]

(d) Proof Consider a partition P_n into n subintervals of equal length: $P_n = \{x_0, x_1, \ldots, x_n\}$,

$$x_i = a + \frac{b-a}{n} i, \qquad i = 0, 1, \ldots, n.$$

Then

$$L(f, P_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i), \qquad U(f, P_n) = \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i),$$

$$U(f, P_n) - L(f, P_n) = \frac{b-a}{n} (f(x_0) - f(x_n)) = \frac{b-a}{n} (f(a) - f(b)) \,.$$

Now, let ε be an arbitrary positive number. Choose n so large that

$$\frac{b-a}{n} (f(a) - f(b)) < \varepsilon.$$

Then

$$U(f, P_n) - L(f, P_n) < \varepsilon$$

and Riemann's Criterion for Integrability tells us that f is Riemann integrable on $[a, b]$. $\square$ **[7 marks; for the case of increasing function proof done in lectures]**

(e) Consider the function $f : [-1, 1] \to \mathbb{R}$ defined as follows:

$$f(x) := \begin{cases} 1 & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x > 0. \end{cases}$$

This function is Riemann integrable on $[-1, 1]$ because it is decreasing but it is not continuous because $\lim_{x \to 0} f(x)$ doesn't exist. **[3 marks; unseen exercise]**

Solution to question 6

(a)

Definition 1 *A function $f : (0, 1] \to \mathbb{R}$ is said to be* locally Riemann integrable *over $(0, 1]$ if $\forall a, b \in (0, 1]$ we have $f \in \mathcal{R}[a, b]$. The set of all locally Riemann integrable functions over the interval $(0, 1]$ will be denoted by $\mathcal{R}_{loc}(0, 1]$.*

[2 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval I and not specifically for the interval $(0, 1]$]

(b) We say that f is integrable on $(0, 1]$ in the improper sense if $\lim_{a \to 0^+} \int_a^1 f(x)\, dx$ exists. We will denote this limit by $\int_0^1 f(x)\, dx$. **[4 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval I and not specifically for the interval $(0, 1]$]**

(c) Note that the integrand is continuous on $(0, 1]$, hence locally integrable on $(0, 1]$. So we have to check only integrability at 0. We have

$$\int_a^1 \frac{1}{\sqrt{x}}\, dx = 2 - 2\sqrt{a} \to 2 \qquad \text{as} \qquad a \to 0^+.$$

[2 marks; special case of an argument done in the lectures]

(d)

Theorem 6 *Let $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d\rangle$, and suppose that*

- $0 \le f(x) \le g(x)$ *for all* $x \in I$*;*
- $\int_c^d g(x)\,dx$ *exists.*

Then $\int_c^d f(x)\,dx$ *exists.*

[3 marks; bookwork]

(e)

Theorem 7 *Suppose that $f \in \mathcal{R}_{loc}(I)$ and $|f|$ is integrable over I. Then f is integrable over I.*

[3 marks; bookwork]

(f) Let us prove the existence of the improper integral

$$\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx\,. \tag{30}$$

Note that the integrand is continuous on $(0, 1]$, hence locally integrable on $(0, 1]$. So we have to check only integrability at 0.

By Theorem 7, in order to prove the existence of the integral (30) it is sufficient to prove the existence of the integral

$$\int_0^1 \frac{\left|\sin\frac{1}{x}\right|}{\sqrt{x}}\,dx\,. \tag{31}$$

Compare now the functions $\dfrac{\left|\sin\frac{1}{x}\right|}{\sqrt{x}}$ and $\dfrac{1}{\sqrt{x}}$ on the interval $(0, 1]$. We have

$$0 \le \frac{\left|\sin\frac{1}{x}\right|}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$$

so, by Theorem 6, in order to prove the existence of the integral (31) it is sufficient to prove the existence of the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. But this has already been done in (c). **[6 marks; unseen exercise]**

(g) Note that the integrand is continuous on $(0, 1]$, hence locally integrable on $(0, 1]$. So we have to check only integrability at 0.

Integrating by parts we get

$$\int_a^1 \frac{\cos\frac{1}{x}}{x^{3/2}}\,dx \;=\; \sqrt{a}\sin\frac{1}{a} - \sin 1 + \frac{1}{2}\int_a^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx\,.$$

We have $\sqrt{a}\sin\frac{1}{a} - \sin 1 \to -\sin 1$ as $a \to 0^+$, so in order to prove the existence of the original integral we have to prove the existence of the integral $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$. But this has already been done in (f). **[5 marks; unseen exercise]**