All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$.
   (i) Define, in terms of ε and δ, the meaning of the mathematical statement "the function f has limit L at the point x_0".
   (ii) Define, in terms of ε and δ, the meaning of the mathematical statement "the function f is continuous at the point x_0".
   (iii) Define the meaning of the mathematical statement "the function f is differentiable at the point x_0".
   (iv) Prove that if f is differentiable at x_0, then f is continuous at x_0.

   (b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows: $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0. \end{cases}$
   (i) Does $\lim\limits_{x \to 0} f(x)$ exist? Justify your answer. If $\lim\limits_{x \to 0} f(x)$ exists, evaluate it.
   (ii) Is f continuous at the point 0? Justify your answer.
   (iii) Is f differentiable at the point 0? Justify your answer.

2. (a) Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.
   (i) Prove that the power series $\sum\limits_{n=1}^{\infty} n a_n x^n$ has radius of convergence R.
   (ii) Prove that the power series $\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R.
   (iii) Prove that the power series $\sum\limits_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ has radius of convergence R.
   (iv) Prove that the function $f : (-R, R) \to \mathbb{R}$, $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, is differentiable and that $f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$. You may use without proof the fact that

   $$\left| \frac{(x_0 + h)^n - x_0^n}{h} - n x_0^{n-1} \right| \leq \frac{n(n-1)|h|}{2} (|x_0| + |h|)^{n-2},$$

   $\forall x_0, h \in \mathbb{R}$, $h \neq 0$, $\forall n \in \mathbb{N}$, $n \geq 2$.

   (b) Construct a power series with infinite radius of convergence whose sum, $f(x)$, is differentiable on $\mathbb{R}$ and satisfies the equation $f'(x) = \cos(x^2)$ on $\mathbb{R}$ as well as the condition $f(0) = 0$. Justify your construction using the results from (a).

3. (a) State Cauchy's Generalisation of the Mean Value Theorem.

   (b) State and prove l'Hôpital's Rule for finding $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ when $f(c) = g(c) = 0$.

   (c) Evaluate the following limits:

      (i) $\displaystyle \lim_{x\to 0} \frac{2x + x^2 + 2\ln(1-x)}{x^3}$,

      (ii) $\displaystyle \lim_{x\to 0} \frac{\tan^2 x - \arctan^2 x}{x^4}$.

4. (a) State the Mean Value Theorem.

   (b) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a, b) \to \mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

   Given an $x \in (a, 0)$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n x$ for some $\xi \in (x, 0)$.

   (c) Use the result from (b) to prove that $\ln\left(\dfrac{1}{4}\right) = -\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n\,4^n}$.

   $\left[\right.$ *Hint:* you may find it helpful to evaluate $\displaystyle\max_{\xi\in[-\frac{3}{4},0]} \frac{\frac{3}{4}+\xi}{1+\xi}$. $\left.\right]$

5. (a) What is a partition of an interval $[a, b]$?

   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded.

      (i) Define the lower and upper Darboux sums of f with respect to a given partition of the interval $[a, b]$

      (ii) Define the lower and upper Riemann integrals of f over $[a, b]$.

      (iii) Define what it means for f to be Riemann integrable on $[a, b]$.

      (iv) State Riemann's Criterion for Integrability.

   (c) Define what it means for a function $f : [a, b] \to \mathbb{R}$ to be decreasing.

   (d) Prove that a decreasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. Here you may use without proof Riemann's Criterion for Integrability.

   (e) Use (d) to provide an example of a bounded function $f : [-1, 1] \to \mathbb{R}$ which is not continuous on $[-1, 1]$ but is Riemann integrable on $[-1, 1]$.

6. (a) Define what it means for a function $f : (0,1] \to \mathbb{R}$ to be locally Riemann integrable.

   (b) Let $f : (0,1] \to \mathbb{R}$ be locally Riemann integrable. Define what it means for f to be integrable on $(0,1]$ in the improper sense.

   (c) Use the definition from (b) to prove the existence of the improper integral

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx\,.$$

   (d) State the Comparison Theorem for Improper Integrals.

   (e) State the theorem relating the integrability, in the improper sense, of the functions f and $|f|$.

   (f) Prove the existence of the improper integral $\displaystyle\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\, dx\,.$

   [*Hint:* you may find it helpful to use the results from (c), (d) and (e).]

   (g) Prove the existence of the improper integral $\displaystyle\int_0^1 \frac{\cos\frac{1}{x}}{x^{3/2}}\, dx\,.$

   [*Hint:* you may find it helpful to integrate by parts.]