MATH1102 Analysis

Solutions to examination paper for the academic year 2010/2011

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Solution to question 1

(a) The function f is said to be *differentiable* at the point x_0 if

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

exists. In this case the limit is denoted by $f'(x_0)$ (or $\frac{df}{dx}(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$) and is called the *derivative* of f at x_0. **[3 marks; bookwork]**

(b)

$$\begin{aligned}
f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{\sqrt{x_0+h}-\sqrt{x_0}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{x_0+h}-\sqrt{x_0})(\sqrt{x_0+h}+\sqrt{x_0})}{h(\sqrt{x_0+h}+\sqrt{x_0})} = \lim_{h\to 0} \frac{(\sqrt{x_0+h})^2-(\sqrt{x_0})^2}{h(\sqrt{x_0+h}+\sqrt{x_0})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x_0+h}+\sqrt{x_0}} \overset{\text{by the Algebra of Limits}}{=} \frac{1}{\lim_{h\to 0}\sqrt{x_0+h}+\sqrt{x_0}} \\
&\qquad \overset{\text{by the continuity of } \sqrt{x}}{=} \frac{1}{\sqrt{x_0}+\sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}\,.
\end{aligned}$$

[4 marks; question from an exercise sheet; however, solutions to this question were not assessed so many students may have skipped it]

(c)

Theorem 1 *If f is differentiable at the point x_0, then f is continuous at the point x_0.*

[3 marks; bookwork]

Proof Let f be differentiable at the point x_0. Then we have

$$0 = f'(x_0)\cdot 0 = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\,\lim_{x\to x_0}(x-x_0) =$$

$$\lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}(x-x_0)\right) = \lim_{x\to x_0}(f(x)-f(x_0)),$$

so

$$\lim_{x \to x_0} (f(x) - f(x_0)) = 0.$$

But we also have the trivial identity

$$\lim_{x \to x_0} f(x_0) = f(x_0).$$

Adding up the latter two identities we get $\lim\limits_{x \to x_0} f(x) = f(x_0)$, which means that the function is continuous at the point x_0. $\square$ **[5 marks; bookwork]**

(d)

Theorem 2 *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable at the point $x_0 \in (a, b)$ and let $g(x_0) \neq 0$. Then $\frac{f}{g}$ is differentiable at x_0 and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2} \ . \tag{1}$$

[3 marks; bookwork]

Proof Let $h \neq 0$ be any number small enough so that $x_0 + h \in (a, b)$ and $g(x_0 + h) \neq 0$; note that the latter can be achieved in view of Theorem 1 and the Inertia Principle. We have

$$\begin{aligned}
\frac{\frac{f(x_0+h)}{g(x_0+h)} - \frac{f(x_0)}{g(x_0)}}{h} &= \frac{f(x_0+h)g(x_0) - f(x_0)g(x_0+h)}{hg(x_0+h)g(x_0)} \\
&= \frac{f(x_0+h)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x_0+h)}{hg(x_0+h)g(x_0)} \\
&= \frac{\frac{f(x_0+h)-f(x_0)}{h} g(x_0) - f(x_0)\,\frac{g(x_0+h)-g(x_0)}{h}}{g(x_0+h)g(x_0)} \ ,
\end{aligned}$$

so by the Algebra of Limits and Theorem 1

$$\begin{aligned}
\lim_{h \to 0} \frac{\frac{f(x_0+h)}{g(x_0+h)} - \frac{f(x_0)}{g(x_0)}}{h} &= \frac{\left(\lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}\right) g(x_0) - f(x_0) \lim\limits_{h \to 0} \frac{g(x_0+h)-g(x_0)}{h}}{\left(\lim\limits_{h \to 0} g(x_0+h)\right) g(x_0)} \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2} \ ,
\end{aligned}$$

which means that the function $\frac{f}{g}$ is differentiable at x_0 and its derivative is given by formula (1). $\square$ **[5 marks; bookwork]**

(e) Using the Quotient Rule, we get

$$\left(\frac{1}{\sqrt{x}}\right)' = \frac{1' \cdot \sqrt{x} - 1 \cdot (\sqrt{x})'}{x} = -\frac{1}{x}(\sqrt{x})'.$$

But, according to (b), $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$, which gives us

$$\left(\frac{1}{\sqrt{x}}\right)' = -\frac{1}{x}\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{2x^{3/2}} \ .$$

[2 marks; special case of an argument done in the lectures]

Solution to question 2

(a)

Definition 1 *We say that the function* $f : [a, b] \to \mathbb{R}$ *achieves a* global maximum *at the point* $c \in [a, b]$ *if* $f(x) \le f(c)\,,\ \ \forall x \in [a, b]$. *The number* $f(c)$ *is called the* maximum value *of* f *and is denoted* $\max\limits_{x\in[a,b]} f(x)$.

Definition 2 *We say that the function* $f : [a, b] \to \mathbb{R}$ *achieves a* global minimum *at the point* $d \in [a, b]$ *if* $f(x) \ge f(d)\,,\ \ \forall x \in [a, b]$. *The number* $f(d)$ *is called the* minimum value *of* f *and is denoted* $\min\limits_{x\in[a,b]} f(x)$.

[3 marks; bookwork]

(b)

Theorem 3 *If* $f :\ [a, b] \to \mathbb{R}$ *is continuous then it achieves its global maximum at some* $c \in [a, b]$ *and its global minimum at some* $d \in [a, b]$.

[3 marks; bookwork]

(c) Suppose $f'(c) > 0$. Consider the function

$$g(x) := \begin{cases} \frac{f(x)-f(c)}{x-c} & \text{if} \quad x \ne c, \\ f'(c) & \text{if} \quad x = c. \end{cases}$$

This function is continuous at the point c, so by the Inertia Principle $\exists\delta > 0$ such that

$$|x - c| < \delta \quad \& \quad x \in [a, b] \qquad \Longrightarrow \qquad g(x) > 0.$$

Choose an arbitrary $\tilde{x} \in [a, b]$ such that $0 < \tilde{x} - c < \delta$. Then

$$g(\tilde{x}) = \frac{f(\tilde{x}) - f(c)}{\tilde{x} - c} > 0$$

which implies $f(\tilde{x}) > f(c)$, contradicting the definition of a global maximum.
Similarly, the assumption $f'(c) < 0$ also leads to a contradiction.
This leaves us with $f'(c) = 0$ as the only possibility. $\square$ **[7 marks; bookwork]**

In the following solutions to (d)(i)–(d)(ii) we denote $f(x) := \frac{x}{\sqrt{2}} + \cos x$.

(d)(i) The unique interior critical point is $\frac{\pi}{4}$ so

$$\max_{x\in[0\,,\,\pi/2]} f(x) = \max\left\{f(0), f\left(\frac{\pi}{4}\right), f\left(\frac{\pi}{2}\right)\right\} = \max\left\{1, \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\,, \frac{\pi}{2\sqrt{2}}\right\} = \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\,.$$

Note that the numerical values of the above three numbers are $\{1\,, 1.26\,, 1.11\}$. Students are expected to be able to establish, without a calculator, that $\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$ is the biggest of the three. The latter statement is equivalent to the inequalities $4(\sqrt{2} - 1) < \pi < 4$ which are more or less obvious. **[7 marks; unseen exercise]**

(d)(ii) We have $f'(x) > 0$, $\forall x \in (-\pi/2\,, 0)$, so f is strictly increasing on $[-\pi/2\,, 0]$ and

$$\max_{x\in[-\pi/2\,,\,0]} f(x) = f(0) = 1.$$

[3 marks; unseen exercise]

(d)(iii) The function $\frac{|x|}{\sqrt{2}}+\cos x$ is even, so

$$\max_{x\in[-\pi/2\,,\,\pi/2]}\left(\frac{|x|}{\sqrt{2}}+\cos x\right)=\max_{x\in[0\,,\,\pi/2]}\left(\frac{x}{\sqrt{2}}+\cos x\right)=\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}\,.$$

[2 marks; unseen exercise]

Solution to question 3

(a) Put $h(x):=f(x)-g(x)$. Then $h'(x)=0$ for all $x\in(-1,1)$. I claim that

$$h(x)=h(0),\qquad \forall x\in(-1,1). \tag{2}$$

Case 1: $x=0$. Obviously, (2) is true.

Case 2: $x\in(0,1)$. Applying the Mean Value Theorem to the function h on the interval $[0,x]$ we get

$$\frac{h(x)-h(0)}{x-0}=h'(\xi) \tag{3}$$

for some $\xi\in(0,x)$. But $h'(\xi)=0$, so (3) implies (2).

Case 3: $x\in(-1,0)$. Applying the Mean Value Theorem to the function h on the interval $[x,0]$ we get

$$\frac{h(0)-h(x)}{0-x}=h'(\xi) \tag{4}$$

for some $\xi\in(x,0)$. But $h'(\xi)=0$, so (4) implies (2). **[8 marks; unseen exercise, but somewhat similar to a lemma proved in the lectures and an exercise from an exercise sheet]**

(b) In the lectures the radius of convergence was defined as follows. Introduce the set

$$S:=\left\{r\ge 0\;\middle|\;\sum_{n=0}^{\infty}a_nx^n\;\text{ converges for } x=r \text{ or } x=-r\right\}. \tag{5}$$

Obviously, $\{0\}\subset S\subset[0,+\infty)$. The *radius of convergence* of a power series is the extended real number

$$R:=\sup S$$

where S is the set (5).

Another possible definition is that the radius of convergence of a power series is the extended real number R such that for $|x|<R$ the series converges absolutely and for $|x|>R$ the series diverges. I will accept this definition even though it is not *a priori* obvious that an R with these properties exists. **[3 marks; bookwork]**

(c) Let $\sum\limits_{n=0}^{\infty}a_nx^n$ be a power series with radius of convergence $R>0$ and let $f:(-R,R)\to\mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R,R)$ and $f'(x)=\sum\limits_{n=1}^{\infty}na_nx^{n-1}$.
[4 marks; bookwork]

(d) Fix an $x\neq 0$ and apply the ratio test:

$$\left|\frac{\frac{(-1)^{k+1}}{2k+3}x^{2k+3}}{\frac{(-1)^k}{2k+1}x^{2k+1}}\right|=\frac{2k+1}{2k+3}x^2=\frac{x^2}{1+\frac{2}{2k+1}}\to x^2$$

as $k \to \infty$. Hence, the power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is 1. [**4 marks; unseen exercise**]

(e) The results from (c) and (d) tell us that the function g is differentiable on $(-1,1)$ and that its derivative is obtained by differentiating the power series for g term-by-term:

$$g'(x) = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}\right)' = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1} x^{2k+1}\right)' = \sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2}$$

because we are looking at a geometric series. [**4 marks; unseen exercise**]

(f) Application of the result from (a) with $f(x) = \arctan x$ and $g(x)$ as in (e) gives

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + c\,, \qquad \forall x \in (-1,1).$$

Setting $x = 0$ we conclude that $c = 0$. [**2 marks; unseen exercise**]

Solution to question 4

(a)

Theorem 4 *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $\xi \in (a,b)$ such that*

$$(g(b) - g(a))f'(\xi) = (f(b) - f(a))g'(\xi)\,. \tag{6}$$

[**4 marks; bookwork**]

(b) Fix $x \in (a,0)$ and consider the function

$$g(t) := f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - \dots - \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x-t)^n\,, \tag{7}$$

where t is the independent variable. Since $f : (a,b) \to \mathbb{R}$ is $n+1$ times differentiable, $g : (a,b) \to \mathbb{R}$ is differentiable. Differentiating (7) we get

$$\begin{aligned} g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \dots \\ \dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] \\ = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n \end{aligned}$$

(all the terms cancelled out, apart from the last one). Applying Cauchy's Generalisation of the Mean Value Theorem to the functions $g(t)$ and $h(t) := (x-t)^{n+1}$ on the interval $[x,0]$ (which we can because g is continuous on $[x,0] \subset (a,b)$ and differentiable on $(x,0) \subset (a,b)$) we get

$$\frac{g(0) - g(x)}{h(0) - h(x)} = \frac{g'(\xi)}{h'(\xi)} = \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{-(n+1)(x-\xi)^n} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi \in (x, 0)$. But

$$g(0) = R_n(x), \quad g(x) = 0, \quad h(0) = x^{n+1}, \quad h(x) = 0,$$

so the above formula can be rewritten as

$$\frac{R_n(x)}{x^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$. $\square$ **[12 marks; bookwork; however in the lectures the above argument was done only for the case $x > 0$]**

(c) Application of the result from (b) to the function $f : (-1, 99) \to \mathbb{R}$, $f(x) = \ln(1 + x)$ with $x = -\frac{1}{2}$ gives

$$\ln\left(\frac{1}{2}\right) = -\sum_{k=1}^{n} \frac{1}{n2^n} + R_n\left(-\frac{1}{2}\right),$$

$$R_n\left(-\frac{1}{2}\right) = (-1)^n \frac{\left(-\frac{1}{2}\right)^{n+1}}{(n+1)(1+\xi)^{n+1}} = -\frac{1}{(n+1)(2+2\xi)^{n+1}}$$

for some $\xi \in (-1/2, 0)$. But the fact that $\xi > -1/2$ implies that $2 + 2\xi > 1$ so

$$\left|R_n\left(-\frac{1}{2}\right)\right| = \frac{1}{(n+1)(2+2\xi)^{n+1}} < \frac{1}{(n+1)} \to 0 \qquad \text{as} \qquad n \to \infty$$

so

$$R_n\left(-\frac{1}{2}\right) \to 0 \qquad \text{as} \qquad n \to \infty$$

which gives the required result. **[9 marks; this question is a special case of a question from an exercise sheet; the tricky bit is that I advised students to use Cauchy's form of the remainder when expanding $\ln(1 + x)$ for $x < 0$ whereas in this examination question I ask them to use Lagrange's form of the remainder; as shown above, Lagrange's form of the remainder is actually OK for $x \geq -1/2$]**

Solution to question 5

(a)(i) The *lower Darboux sum* of f with respect to the partition P is defined as

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

where $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$. The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i := \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$. **[2 marks; bookwork]**

(a)(ii) The *lower Riemann integral* of f over $[a, b]$ is defined as

$$\underline{\int_a^b} f(x)\, dx := \sup_{P \in \mathcal{P}[a,b]} L(f, P).$$

The *upper Riemann integral* of f over $[a,b]$ is defined as

$$\overline{\int_a^b} f(x)\,dx := \inf_{P\in\mathcal{P}[a,b]} U(f,P)\,.$$

Here $\mathcal{P}[a,b]$ is the set of all partitions of the interval $[a,b]$. **[2 marks; bookwork]**

(a)(iii) We know that $L(f,P) \le U(f,Q)$ for any pair of partitions P and Q. Taking the supremum over $P \in \mathcal{P}[a,b]$ we get

$$\underline{\int_a^b} f(x)\,dx \le U(f,Q). \tag{8}$$

Note that formula (8) holds for any partition Q. Taking the infimum over $Q \in \mathcal{P}[a,b]$ we get

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx\,.$$

[2 marks; bookwork]

(a)(iv)

Definition 3 *The function f is said to be* Riemann integrable *on $[a,b]$ if*

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx\,.$$

In this case the common value of the lower and upper Riemann integrals is denoted by

$$\int_a^b f(x)\,dx$$

and is called the Riemann integral *of f over $[a,b]$. The set of all Riemann integrable functions $f:[a,b]\to\mathbb{R}$ will be denoted by $\mathcal{R}[a,b]$.*

[2 marks; bookwork]

(b)

Theorem 5 *Let $f:[a,b]\to\mathbb{R}$ be bounded. Then $f\in\mathcal{R}[a,b]$ iff $\forall\varepsilon>0\ \ \exists P\in\mathcal{P}[a,b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{9}$$

[3 marks; bookwork]

Proof

Part 1. Suppose $\forall\varepsilon>0\ \exists P\in\mathcal{P}[a,b]$ such that (9) holds. Need to prove that $f\in\mathcal{R}[a,b]$.

Let ε be an arbitrary positive number. Choose $P\in\mathcal{P}[a,b]$ such that (9) holds. Then

$$U(f,P)-\varepsilon < L(f,P) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(f,P),$$

so

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon.$$

As $\varepsilon>0$ is arbitrary, this implies

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0$$

which means that $f \in \mathcal{R}[a,b]$.

Part 2. Suppose $f \in \mathcal{R}[a,b]$. Need to prove that $\forall \varepsilon > 0\ \exists P \in \mathcal{P}[a,b]$ such that (9) holds.

Let ε be an arbitrary positive number. Choose $P', P'' \in \mathcal{P}[a,b]$ such that

$$\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < L(f,P'), \tag{10}$$

$$U(f,P'') < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}. \tag{11}$$

Let $P := P' \cup P''$ be a common refinement of P' and P''. Then, by the refinement lemma,

$$L(f,P') \le L(f,P), \qquad U(f,P) \le U(f,P'').$$

But, obviously, $L(f,P) \le U(f,P)$, so

$$L(f,P') \le L(f,P) \le U(f,P) \le U(f,P''). \tag{12}$$

Formulae (10)–(12) imply

$$\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < L(f,P) \le U(f,P) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

which gives (9). $\square$ **[7 marks; bookwork]**

(c) Let ε be an arbitrary positive number. We know (MATH1101) that continuity of the function $f : [a,b] \to \mathbb{R}$ implies its uniform continuity, so $\exists \delta > 0$ such that

$$|x-y| < \delta \quad \& \quad x, y \in [a,b] \qquad \Longrightarrow \qquad |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}\,.$$

Choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ with $\|P\| < \delta$. Then

$$M_i - m_i \le \frac{\varepsilon}{2(b-a)}\,;$$

here we use the usual notation $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$, $M_i := \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$. We get

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \le \frac{\varepsilon}{2(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\varepsilon}{2} < \varepsilon$$

and Riemann's Criterion for Integrability (Theorem 5) tells us that $f \in \mathcal{R}[a,b]$. $\square$ **[4 marks; bookwork]**

(d) Consider the function $f : [-1,1] \to \mathbb{R}$ defined as follows:

$$f(x) := \begin{cases} -1 & \text{if} \quad x < 0, \\ \ \ 0 & \text{if} \quad x = 0, \\ \ \ 1 & \text{if} \quad x > 0. \end{cases}$$

This function is Riemann integrable on $[-1,1]$ because it is monotone (increasing) but it is not continuous because $\lim\limits_{x \to 0} f(x)$ doesn't exist. **[3 marks; unseen exercise]**

Solution to question 6

(a)

Definition 4 *A function* $f : [1, +\infty) \to \mathbb{R}$ *is said to be* locally Riemann integrable *over* $[1, +\infty)$ *if* $\forall a, b \in [1, +\infty)$ *we have* $f \in \mathcal{R}[a, b]$. *The set of all locally Riemann integrable functions over the interval* $[1, +\infty)$ *will be denoted by* $\mathcal{R}_{loc}[1, +\infty)$.

[2 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval I **and not specifically for the interval** $[1, +\infty)$**]**

(b) We say that f is integrable on $[1, +\infty)$ in the improper sense if $\lim\limits_{b\to+\infty} \int_1^b f(x)\,dx$ exists. We will denote this limit by $\int_1^{+\infty} f(x)\,dx$. **[4 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval** I **and not specifically for the interval** $[1, +\infty)$**]**

(c) Note that the integrand is continuous on $[1, +\infty)$, hence locally integrable on $[1, +\infty)$. So we have to check only integrability at $+\infty$. We have

$$\int_1^b \frac{1}{x^2}\,dx = 1 - \frac{1}{b} \to 1 \qquad \text{as} \qquad b \to +\infty.$$

[2 marks; special case of an argument done in the lectures]

(d)

Theorem 6 *Let* $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d \rangle$, *and suppose that*

- $0 \le f(x) \le g(x)$ *for all* $x \in I$;
- $\int_c^d g(x)\,dx$ *exists.*

Then $\int_c^d f(x)\,dx$ *exists.*

[3 marks; bookwork]

(e)

Theorem 7 *Suppose that* $f \in \mathcal{R}_{loc}(I)$ *and* $|f|$ *is integrable over* I. *Then* f *is integrable over* I.

[3 marks; bookwork]

(f) Let us prove the existence of the improper integral

$$\int_1^{+\infty} \frac{\sin x}{x^2}\,dx\,. \tag{13}$$

Note that the integrand is continuous on $[1, +\infty)$, hence locally integrable on $[1, +\infty)$. So we have to check only integrability at $+\infty$.

By the result from (e), in order to prove the existence of the integral (13) it is sufficient to prove the existence of the integral

$$\int_1^{+\infty} \frac{|\sin x|}{x^2}\,dx\,. \tag{14}$$

Compare now the functions $\dfrac{|\sin x|}{x^2}$ and $\dfrac{1}{x^2}$ on the interval $[1, +\infty)$. We have

$$0 \leq \frac{|\sin x|}{x^2} \leq \frac{1}{x^2}$$

so, by Theorem 6, in order to prove the existence of the integral (14) it is sufficient to prove the existence of the integral $\int_1^{+\infty} \frac{1}{x^2}\,dx$. But this has already been done in (c). **[6 marks; unseen exercise]**

(g) Note that the integrand is continuous on $[1, +\infty)$, hence locally integrable on $[1, +\infty)$. So we have to check only integrability at $+\infty$.

Integrating by parts we get

$$\int_1^b \frac{\cos x}{x}\,dx = \frac{\sin b}{b} - \sin 1 + \int_1^b \frac{\sin x}{x^2}\,dx\,.$$

We have $\frac{\sin b}{b} - \sin 1 \to -\sin 1$ as $b \to +\infty$, so in order to prove the existence of the original integral we have to prove the existence of the integral $\int\limits_1^{+\infty} \frac{\sin x}{x^2}\,dx$. But this has already been done in (f). **[5 marks; question from an exercise sheet; however, this is the last question of the last exercise sheet and, moreover, solutions to this question were not assessed so many students may have skipped it]**