All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Consider a function $f : (0, +\infty) \to \mathbb{R}$ and a point $x_0 \in (0, +\infty)$. Define the meaning of the statement "the function f is differentiable at the point x_0".

   (b) Consider the function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ and a point $x_0 \in (0, +\infty)$. Use the definition from (a) to prove that this function is differentiable at the point x_0 and that $f'(x_0) = \dfrac{1}{2\sqrt{x_0}}$. Here you may use without proof the fact that the function $\sqrt{x}$ is continuous.

   (c) State and prove the theorem about the relationship between differentiability and continuity.

   (d) State and prove the Quotient Rule.

   (e) Consider the function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{\sqrt{x}}$. Prove that this function is differentiable and that $f'(x) = -\dfrac{1}{2x^{3/2}}$.

2. (a) Define what it means for a function $f : [a, b] \to \mathbb{R}$ to achieve a *global maximum* at a point $c \in [a, b]$ and a *global minimum* at a point $d \in [a, b]$.

   (b) State the Attainment of Bounds Theorem.

   (c) Suppose that the function $f : [a, b] \to \mathbb{R}$ is differentiable at the point $c \in (a, b)$ and suppose that f achieves a global maximum at the point c. Prove that $f'(c) = 0$.

   (d) Find, with justification,

      (i) $\max\limits_{x \in [0\,,\,\pi/2]} \left(\dfrac{x}{\sqrt{2}} + \cos x \right)$,

      (ii) $\max\limits_{x \in [-\pi/2\,,\,0]} \left(\dfrac{x}{\sqrt{2}} + \cos x \right)$,

      (iii) $\max\limits_{x \in [-\pi/2\,,\,\pi/2]} \left(\dfrac{|x|}{\sqrt{2}} + \cos x \right)$.

3. (a) Suppose that the functions $f, g : (-1, 1) \to \mathbb{R}$ are differentiable and that $f'(x) = g'(x)$ for all $x \in (-1, 1)$. Prove that $f(x) = g(x) + c$ for all $x \in (-1, 1)$, where c is a constant.

   (b) Define the notion of the radius of convergence of a power series.

   (c) State the theorem about the differentiability of power series.

   (d) Find the radius of convergence of the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$.

   (e) Consider the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. Prove that

$$g'(x) = \frac{1}{1+x^2}.$$

   (f) Prove that $\arctan x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ for all $x \in (-1, 1)$. Here you may use without proof the fact that $\arctan' x = \dfrac{1}{1+x^2}$.

   [*Hint:* you may find it helpful to use the results from (a) and (e).]

4. (a) State Cauchy's Generalisation of the Mean Value Theorem.

   (b) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a, b) \to \mathbb{R}$ be $n + 1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

   Given an $x \in (a, 0)$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$ for some $\xi \in (x, 0)$.

   (c) Use the result from (b) to prove that $\ln\left(\dfrac{1}{2}\right) = -\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

5. (a) Let $f : [a,b] \to \mathbb{R}$ be bounded.

   (i) Define the lower Darboux sum $L(f,P)$ and the upper Darboux sum $U(f,P)$ of f with respect to a given partition P of the interval $[a,b]$.

   (ii) Define the lower Riemann integral $\underline{\int_a^b} f(x)\,dx$ and the upper Riemann integral $\overline{\int_a^b} f(x)\,dx$.

   (iii) Prove that $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$. Here you may use without proof the fact that $L(f,P) \leq U(f,Q)$ for any pair of partitions P and Q.

   (iv) Define what it means for f to be Riemann integrable on $[a,b]$.

   (b) State and prove Riemann's Criterion for Integrability. Here you may use without proof the refinement lemma: if the partition P' is a refinement of the partition P then $L(f,P) \leq L(f,P')$ and $U(f,P) \geq U(f,P')$.

   (c) Let $f : [a,b] \to \mathbb{R}$ be continuous. Prove that f is Riemann integrable on $[a,b]$.

   (d) Give an example of a bounded function $f : [-1,1] \to \mathbb{R}$ which is not continuous on $[-1,1]$ but is Riemann integrable on $[-1,1]$. Justify your answer. You may use any result from the course.

6. (a) Define what it means for a function $f : [1,+\infty) \to \mathbb{R}$ to be locally Riemann integrable.

   (b) Let $f : [1,+\infty) \to \mathbb{R}$ be locally Riemann integrable. Define what it means for f to be integrable on $[1,+\infty)$ in the improper sense.

   (c) Use the definition from (b) to prove the existence of the improper integral

$$\int_1^{+\infty} \frac{1}{x^2}\,dx\,.$$

   (d) State the Comparison Theorem for Improper Integrals.

   (e) State the theorem relating the integrability, in the improper sense, of the functions f and $|f|$.

   (f) Prove the existence of the improper integral $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,dx\,.$

   [*Hint:* you may find it helpful to use the results from (c), (d) and (e).]

   (g) Prove the existence of the improper integral $\displaystyle\int_1^{+\infty} \frac{\cos x}{x}\,dx\,.$

   [*Hint:* you may find it helpful to integrate by parts.]