MATH1102 Analysis

Solutions to examination paper for the academic year 2009/2010

Dmitri Vassiliev

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Solution to question 1

(a)(i) We say that the function $f : D \to \mathbb{R}$ is *continuous* at the point x_0 if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

[3 marks; bookwork]

(a)(ii) The function f is said to be *differentiable* at the point x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In this case the limit is denoted by $f'(x_0)$ (or $\frac{df}{dx}(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$) and is called the *derivative* of f at x_0. **[3 marks; bookwork]**

(a)(iii)

Theorem 1 *If f is differentiable at the point x_0, then f is continuous at the point x_0.*

[3 marks; bookwork]

(b)

Theorem 2 *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable at the point $x_0 \in (a, b)$. Then fg is differentiable at x_0 and*

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0). \tag{1}$$

[3 marks; bookwork]

Proof Let $h \neq 0$ be any number small enough so that $x_0 + h \in (a, b)$. We have

$$\frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0)}{h}$$
$$= \frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0 + h) + f(x_0)g(x_0 + h) - f(x_0)g(x_0)}{h}$$
$$= \frac{f(x_0 + h) - f(x_0)}{h}\, g(x_0 + h) + f(x_0)\, \frac{g(x_0 + h) - g(x_0)}{h} \,,$$

so by the Algebra of Limits and Theorem 1

$$\lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h}$$
$$= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \lim_{h\to 0} g(x_0+h) + f(x_0) \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h}$$
$$= f'(x_0)\, g(x_0) \;+\; f(x_0)\, g'(x_0)\,,$$

which means that the function fg is differentiable at x_0 and its derivative is given by formula (1). $\square$ **[7 marks; bookwork]**

(c) We need to prove that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = 0.$$

That is, we need to prove that

$$\lim_{x\to 0} x^3 \cos(x^{-2}) = 0.$$

In order to prove the above statement, we apply the ε, δ definition of a limit. Let ε be an arbitrary positive number. We need to find a $\delta > 0$ such that

$$0 < |x| < \delta \qquad \Longrightarrow \qquad \left| x^2 \cos(x^{-2}) \right| < \varepsilon.$$

As $\left|\cos(x^{-2})\right| \le 1$, we see that $\delta = \sqrt{\varepsilon}$ would do. **[6 marks; unseen exercise but similar to a question in an exercise sheet]**

Solution to question 2

(a)(i) Consider two power series:

$$\sum_{n=0}^{\infty} a_n x^n \tag{2}$$

and

$$\sum_{n=0}^{\infty} n a_n x^n. \tag{3}$$

Lemma 1 *The two power series (2) and (3) have the same radii of convergence.*

Proof Denote the radii of convergence of the power series (2) and (3) by R and R' respectively. Let us prove first that

$$R' \le R. \tag{4}$$

Suppose (4) is false. Then $R' > R$ and we can choose an x such that $R < x < R'$. For this x the series (3) converges absolutely whereas the series (2) diverges. But

$$|a_n x^n| \le |n a_n x^n|$$

for $n \ge 1$ so the series (2) converges by the comparison test. Thus, the series (2) diverges and converges, which is impossible. This contradiction proves (4).

Let us prove now that

$$R \le R'. \tag{5}$$

Suppose (5) is false. Then $R > R'$ and we can choose y, z such that $R' < y < z < R$. For these y and z the series

$$\sum_{n=0}^{\infty} a_n z^n \tag{6}$$

converges absolutely whereas the series

$$\sum_{n=0}^{\infty} n a_n y^n \tag{7}$$

diverges. But

$$|n a_n y^n| = |a_n z^n| \left(n \left| \frac{y}{z} \right|^n \right). \tag{8}$$

It is known[1] from MATH1101 that $\lim_{n\to\infty} \left(n \left| \frac{y}{z} \right|^n \right) = 0$ so there exists an $M \geq 0$ such that $n \left| \frac{y}{z} \right|^n \leq M$, $\forall n \in \mathbb{N} \cup \{0\}$, and formula (8) implies $|n a_n y^n| \leq M |a_n z^n|$. Hence the series (7) converges by the comparison test. Thus, the series (7) diverges and converges, which is impossible. This contradiction proves (5).

Formulae (4) and (5) imply $R = R'$. $\square$ **[10 marks; bookwork]**

(a)(ii) Now consider the power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1}. \tag{9}$$

Note that (9) is the formal derivative of the original power series (2).

Lemma 2 *The two power series (2) and (9) have the same radii of convergence.*

Proof The two power series obviously converge for $x = 0$. For $x \neq 0$ we have

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} n a_n x^n$$

and the required result follows from Lemma 1. $\square$ **[2 marks; bookwork]**

(a)(iii) Now consider the power series

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \tag{10}$$

where k is an arbitrary fixed nonnegative integer number. Note that (9) is the formal kth derivative of the original power series (2).

Lemma 3 *The two power series (2) and (10) have the same radii of convergence.*

Proof Repeated application of Lemma 2. $\square$ **[1 mark; bookwork]**

(a)(iv)

Theorem 3 *Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$.*

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[1] Here I use the standard fact that $\lim_{n\to\infty} n\alpha^n = 0$ for $0 < \alpha < 1$. Indeed, set $\alpha = \frac{1}{1+\beta}$. Then $\beta > 0$ and $n\alpha^n = \dfrac{n}{(1+\beta)^n} \leq \dfrac{n}{\frac{n(n-1)}{2}\beta^2} = \dfrac{2}{\beta^2(n-1)} \to 0.$

Proof Fix an $x_0 \in (-R, R)$ and an h_0 such that $0 < h_0 < R - |x_0|$ and let h be such that $0 < |h| < h_0$. Then by the results from (a)(ii) and (a)(iii) and the inequality stated in the examination paper

$$\left| \frac{\sum\limits_{n=0}^{\infty} a_n (x_0 + h)^n - \sum\limits_{n=0}^{\infty} a_n x_0^n}{h} - \sum_{n=1}^{\infty} n a_n x_0^{n-1} \right| = \left| \sum_{n=2}^{\infty} a_n \left(\frac{(x_0+h)^n - x_0^n}{h} - n x_0^{n-1} \right) \right|$$

$$\leq \sum_{n=2}^{\infty} \left| a_n \left(\frac{(x_0+h)^n - x_0^n}{h} - n x_0^{n-1} \right) \right| \leq \frac{|h|}{2} \sum_{n=2}^{\infty} n(n-1)|a_n|(|x_0| + |h|)^{n-2}$$

$$\leq \underbrace{\frac{|h|}{2} \sum_{n=2}^{\infty} n(n-1)|a_n|(|x_0| + h_0)^{n-2}}_{\text{number, not a function of } h} \to 0$$

as $h \to 0$, giving the required result. Note that the results from (a)(ii) and (a)(iii) play an important role in this proof: they guarantee that all the series appearing in the above argument converge absolutely. $\square$ **[7 marks; bookwork]**

(b) In view of the result from (a)(iv) it is sufficient to show that the power series

$$\sum_{n=1}^{\infty} n x^{n-1} \tag{11}$$

has radius of convergence 1. But this power series is the formal derivative of the power series

$$\sum_{n=0}^{\infty} x^n \tag{12}$$

which is the geometric series and is known to have radius of convergence 1. The result from (a)(ii) tells us that the power series (11) also has radius of convergence 1. **[5 marks; unseen exercise]**

Solution to question 3

(a)

Theorem 4 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and let $f(a) = f(b)$. Then there exists a point $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

[3 marks; bookwork]

Proof According to the Attainment of Bounds Theorem there exist $c, d \in [a, b]$ such that

$$f(d) \leq f(x) \leq f(c), \qquad \forall x \in [a, b].$$

If both c and d are endpoints of our interval then $f(x) = \text{const}$ and $f'(x) = 0$, $\forall x \in (a, b)$, so any $\xi \in (a, b)$ would do. If c is not an endpoint then, by the fact stated in the examination paper, $f'(c) = 0$, so we can take $\xi = c$. If d is not an endpoint then, by the fact stated in the examination paper, $f'(d) = 0$, so we can take $\xi = d$. $\square$ **[6 marks; bookwork]**

(b)

Theorem 5 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}\,. \tag{13}$$

[3 marks; bookwork]

Proof Let us introduce a new function $g(x) = f(x) - kx$ where the constant k is chosen so that the function g satisfies the conditions of Rolle's Theorem:

$$g(a) = g(b) \iff f(a) - ka = f(b) - kb \iff k = \frac{f(b) - f(a)}{b - a}.$$

Thus, the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a} x$$

satisfies the conditions of Rolle's Theorem. Rolle's Theorem tell us that there exists a point $\xi \in (a, b)$ such that

$$g'(\xi) = 0 \iff f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0 \iff f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

□ **[5 marks; bookwork]**

(c) If $x = y$ the inequality

$$|\arctan x - \arctan y| \leq |x - y|$$

is certainly true, so further one we consider the case $x \neq y$. Without loss of generality we also assume that $y < x$.

The function $\arctan$ is differentiable (and, therefore, continuous) on the whole real line, hence we can apply the Mean Value Theorem to the function $\arctan$ on the interval $[y, x]$ to get

$$\arctan'(\xi) = \frac{1}{1 + \xi^2} = \frac{\arctan x - \arctan y}{x - y}$$

for some $\xi \in (y, x)$. Hence,

$$|\arctan x - \arctan y| = \frac{|x - y|}{1 + \xi^2} \leq |x - y|.$$

[5 marks; unseen exercise]

(d)

Theorem 6 *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $\xi \in (a, b)$ such that*

$$(g(b) - g(a))f'(\xi) = (f(b) - f(a))g'(\xi). \tag{14}$$

[3 marks; bookwork]

Solution to question 4

(a)

Theorem 7 *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable and let $c \in (a, b)$ be such that $f(c) = g(c) = 0$ and $g'(x) \neq 0$ for $x \neq c$. Then*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{15}$$

provided the latter limit exists.

[5 marks; bookwork]

Explanation why $g(x) \neq 0$ for $x \neq c$ (this is not part of the proof of the theorem). Indeed, suppose $g(x) = 0$ for some $x \in (a, b)$, $x \neq c$.

Case 1: $x > c$. Apply Rolle's Theorem to the function g on the interval $[c, x]$: Rolle's Theorem tells us that there exists a $\xi \in (c, x)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 7.

Case 2: $x < c$. Apply Rolle's Theorem to the function g on the interval $[x, c]$: Rolle's Theorem tells us that there exists a $\xi \in (x, c)$ such that $g'(\xi) = 0$, which contradicts the assumptions of Theorem 7.

Proof of Theorem 7 Suppose $\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists and equals L. Then

$$\lim_{x\to c+} \frac{f'(x)}{g'(x)} = \lim_{x\to c-} \frac{f'(x)}{g'(x)} = L.$$

In order to prove (15) we need to show that

$$\lim_{x\to c+} \frac{f(x)}{g(x)} = L \tag{16}$$

and

$$\lim_{x\to c-} \frac{f(x)}{g(x)} = L. \tag{17}$$

Let us prove (16). Take an arbitrary sequence $\{x_n\} \subset (c, b)$ such that

$$\lim_{n\to\infty} x_n = c. \tag{18}$$

Applying Cauchy's Generalisation of the Mean Value Theorem on the interval $[c, x_n]$ we get a sequence $\{\xi_n\}$ such that

$$c < \xi_n < x_n \tag{19}$$

and

$$(g(x_n) - g(c))f'(\xi_n) = (f(x_n) - f(c))g'(\xi_n)$$

which can be equivalently rewritten as

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(\xi_n)}{g'(\xi_n)}. \tag{20}$$

Formulae (18), (19) and the Sandwich Principle imply

$$\lim_{n\to\infty} \xi_n = c. \tag{21}$$

Formula (21) and the sequential definition of a limit (one-sided version) imply

$$\lim_{n\to\infty} \frac{f'(\xi_n)}{g'(\xi_n)} = L. \tag{22}$$

Formulae (20) and (22) imply

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L. \tag{23}$$

Formula (23) and the sequential definition of a limit (one-sided version) imply (16).

Formula (17) is proved similarly. $\square$ **[9 marks; bookwork]**

(b)(i) Using L'Hôpital's Rule twice we get

$$\lim_{x\to -2} \frac{x^3 + 6x^2 + 12x + 8}{x^3 + 5x^2 + 8x + 4} = \lim_{x\to -2} \frac{3x^2 + 12x + 12}{3x^2 + 10x + 8} = \lim_{x\to -2} \frac{6x + 12}{6x + 10} = \frac{6 \times (-2) + 12}{6 \times (-2) + 10} = 0.$$

[3 marks; unseen exercise]

(b)(ii) Using L'Hôpital's Rule once we get

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{x\to\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin x + \left(\frac{\pi}{2} - x\right) \cos x}{-\sin x} = 1$$

so

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^2 \tan^2 x = \left(\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x\right)^2 = 1.$$

[3 marks; unseen exercise]

(b)(iii) We have

$$\begin{aligned} \frac{\tan^3 x - \sin^3 x}{x^5} &= \left(\frac{\sin x}{x}\right)^3 \frac{\frac{1}{\cos x} - 1}{x^2} \left(\frac{1}{\cos^2 x} + \frac{1}{\cos x} + 1\right) \\ &= \left(\frac{\sin x}{x}\right)^3 \frac{1 - \cos x}{x^2} \left(\frac{1}{\cos^3 x} + \frac{1}{\cos^2 x} + \frac{1}{\cos x}\right). \end{aligned} \tag{24}$$

It is a standard fact (proved by applying L'Hôpital's Rule once) that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{25}$$

By the continuity of the function cos we have

$$\lim_{x\to 0} \left(\frac{1}{\cos^3 x} + \frac{1}{\cos^2 x} + \frac{1}{\cos x}\right) = \frac{1}{\cos^3 0} + \frac{1}{\cos^2 0} + \frac{1}{\cos 0} = 3. \tag{26}$$

Using L'Hôpital's Rule twice we get

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{\cos 0}{2} = \frac{1}{2}. \tag{27}$$

Combining formulae (24)–(27) we get

$$\lim_{x\to 0} \frac{\tan^3 x - \sin^3 x}{x^5} = \frac{3}{2}\,.$$

[5 marks; unseen exercise]

Solution to question 5

(a)

Definition 1 *Let $f : I \to \mathbb{R}$. A* primitive *of f is a function $F : I \to \mathbb{R}$ which is continuous on I, differentiable on the interior of I and satisfies $F'(x) = f(x)$ on the interior of I.*

[2 marks; bookwork]

(b) No: one can always add a constant to F. **[1 mark; bookwork]**

(c)

Theorem 8 *Let $f \in \mathcal{R}[a, b]$ and let F be a primitive of f. Then*

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a) \;=\; F(x)\big|_a^b\,.$$

[4 marks; bookwork]

Note: $F(x)|_a^b$ is simply a short way of writing $F(b) - F(a)$.

Proof of Theorem 8 Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Applying the Mean Value Theorem to the function F on the subinterval $[x_{i-1}, x_i]$ we get

$$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1})$$

for some $t_i \in (x_{i-1}, x_i)$. But

$$m_i \leq f(t_i) \leq M_i$$

where I use the standard notation $m_i := \inf_{x\in[x_{i-1},x_i]} f(x)$, $M_i := \sup_{x\in[x_{i-1},x_i]} f(x)$. Hence,

$$m_i(x_i - x_{i-1}) \leq F(x_i) - F(x_{i-1}) \leq M_i(x_i - x_{i-1}).$$

Summing up over i from 1 to n we get

$$L(f, P) \leq \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \leq U(f, P).$$

But

$$\sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(x_n) - F(x_0) = F(b) - F(a),$$

so the above inequality can be rewritten as

$$L(f, P) \leq F(b) - F(a) \leq U(f, P). \tag{28}$$

The inequality (28) is true for any partition P. Taking a limiting sequence of partitions $\{P_n\}$ and letting $n \to \infty$ we get from (28)

$$\lim_{n\to\infty} L(f, P_n) \leq F(b) - F(a) \leq \lim_{n\to\infty} U(f, P_n).$$

It remains to note that as $f \in \mathcal{R}[a, b]$ the Theorem about limiting sequences tells us that

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \int_a^b f(x)\, dx\,.$$

$\square$ **[8 marks; bookwork]**

(d)

Definition 2 *Let $f \in \mathcal{R}_{loc}(I)$. An* indefinite integral *of f is a function $F : I \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\, dt$$

for some $a \in I$.

[2 marks; bookwork]

(e)

Theorem 9 *Let $f \in \mathcal{R}_{loc}(I)$ and let $F : I \to \mathbb{R}$ be an indefinite integral of f. Then*

(i) F is continuous on I;

(ii) F is differentiable at each interior point $x_0 \in I$ at which f is continuous, and at such a point $F'(x_0) = f(x_0)$;

(iii) if f is continuous on I then F is a primitive of f.

[4 marks; bookwork]

(f) We have
$$G(x) = F(\sin x) - F(\cos x)$$
where
$$F(y) = \int_0^y e^{t^2}\, dt\,.$$
Hence, by the Chain Rule and the Theorem about the Properties of the Indefinite Integral we have
$$G'(x) = F'(\sin x)\cos x + F'(\cos x)\sin x = e^{\sin^2 x}\cos x + e^{\cos^2 x}\sin x.$$

[4 marks; unseen exercise]

Solution to question 6

(a)

Definition 3 *A function $f : I \to \mathbb{R}$ is said to be* locally Riemann integrable *over I if $\forall a, b \in I$ we have $f \in \mathcal{R}[a, b]$. The set of all locally Riemann integrable functions over the interval I will be denoted by $\mathcal{R}_{loc}(I)$.*

[2 marks; bookwork]

(b)

Definition 4 *Let $f \in \mathcal{R}_{loc}(I)$, $I = \langle c, d\rangle$.*

(i) We say that f is integrable at c in the improper sense if $\exists p \in I$, $c < p$, such that $\lim\limits_{a\to c+} \int\limits_a^p f(x)\, dx$ exists. We will denote this limit by $\int_c^p f(x)\, dx$.

(ii) We say that f is integrable at d in the improper sense if $\exists p \in I$, $p < d$, such that $\lim\limits_{b\to d-} \int\limits_p^b f(x)\, dx$ exists. We will denote this limit by $\int_p^d f(x)\, dx$.

(iii) We say that f is integrable over I in the improper sense if both (i) and (ii) hold for the same p. We will then write $\int\limits_c^d f(x)\, dx := \int\limits_c^p f(x)\, dx + \int\limits_p^d f(x)\, dx$.

[5 marks; bookwork]

(c)

Theorem 10 *Let $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d\rangle$, and suppose that*

- *$0 \le f(x) \le g(x)$ for all $x \in I$;*
- *$\int_c^d g(x)\, dx$ exists.*

Then $\int_c^d f(x)\,dx$ *exists.*

[3 marks; bookwork]

Proof Given arbitrary $a, b \in I$, $a < b$, we have

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx\,. \tag{29}$$

But according to the fact stated in the examination paper there exists a K such that

$$\int_a^b g(x)\,dx \ \le K \tag{30}$$

irrespective of the choice of a, b. Combining formulae (29) and (30) we get

$$\int_a^b f(x)\,dx \ \le K$$

irrespective of the choice of a, b. But according to the fact stated in the examination paper this implies that $\int_c^d f(x)\,dx$ exists. $\square$ **[5 marks; bookwork]**

(d) Introduce the functions

$$f^+(x) := \frac{|f(x)| + f(x)}{2}\,, \qquad f^-(x) := \frac{|f(x)| - f(x)}{2}\,.$$

It is easy to see that

$$0 \le f^+(x) \le |f(x)|, \tag{31}$$

$$0 \le f^-(x) \le |f(x)| \tag{32}$$

(to check the above inequalities consider separately the cases $f(x) \ge 0$ and $f(x) < 0$). By the Theorem about the Properties of the Riemann Integral we have $f^+, f^- \in \mathcal{R}_{loc}(I)$. Formula (31) and the Comparison Theorem for Improper Integrals (Theorem 10) imply that f^+ is integrable over I. Similarly, formula (32) implies that f^- is integrable over I. But $f(x) = f^+(x) - f^-(x)$, so the integrability of f over I follows by linearity. $\square$ **[4 marks; bookwork]**

(e) Let us prove the existence of the improper integral

$$\int_0^\infty \frac{\sin x}{1+x^4}\,dx\,. \tag{33}$$

As there is no problem at $x = 0$, in order to prove the existence of the integral (33) it is sufficient to prove the existence of the integral

$$\int_1^\infty \frac{\sin x}{1+x^4}\,dx\,. \tag{34}$$

By the result from (d), in order to prove the existence of the integral (34) it is sufficient to prove the existence of the integral

$$\int_1^\infty \frac{|\sin x|}{1+x^4}\,dx\,. \tag{35}$$

Compare now the functions $\dfrac{|\sin x|}{1+x^4}$ and $\dfrac{1}{x^4}$ on the interval $[1, \infty)$ We have

$$0 \le \frac{|\sin x|}{1+x^4} \le \frac{1}{x^4}$$

so, by Theorem 10, in order to prove the existence of the integral (35) it is sufficient to prove the existence of the integral

$$\int_1^\infty \frac{1}{x^4}\,dx\,. \tag{36}$$

But the existence of the integral (36) is a standard fact. **[6 marks; unseen exercise]**