All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$.
   (i) Define, in terms of ε and δ, the meaning of the statement "the function f is continuous at the point x_0".
   (ii) Define the meaning of the statement "the function f is differentiable at the point x_0".
   (iii) State the theorem about the relationship between differentiability and continuity.

   (b) State and prove the Product Rule.

   (c) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows: $f(x) = \begin{cases} x^3 \cos(x^{-2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
   Prove that f is differentiable at 0 and that $f'(0) = 0$. Here you may use without proof all the standard properties of trigonometric functions.
   [*Hint:* think carefully which rules, if any, might it be useful to apply.]

2. (a) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.
   (i) Prove that the power series $\sum_{n=1}^{\infty} n a_n x^n$ has radius of convergence R.
   (ii) Prove that the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R.
   (iii) Prove that the power series $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ has radius of convergence R.
   (iv) Prove that the function $f : (-R, R) \to \mathbb{R}$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$, is differentiable and that $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. You may use without proof the fact that
   $$\left| \frac{(x_0 + h)^n - x_0^n}{h} - n x_0^{n-1} \right| \leq \frac{n(n-1)|h|}{2} (|x_0| + |h|)^{n-2},$$
   $\forall x_0, h \in \mathbb{R}$, $h \neq 0$, $\forall n \in \mathbb{N}$, $n \geq 2$.

   (b) Prove that the series $\sum_{n=1}^{\infty} n x^{n-1}$ converges absolutely for $|x| < 1$ and that the function $f : (-1, 1) \to \mathbb{R}$, $f(x) = \sum_{n=1}^{\infty} n x^{n-1}$, is differentiable.

3. (a) State and prove Rolle's Theorem. Here you may use without proof the following fact: if the function $f : [a, b] \to \mathbb{R}$ achieves a global maximum (minimum) at the point $x_0 \in (a, b)$ and if f is differentiable at the point x_0 then $f'(x_0) = 0$.

   (b) State and prove the Mean Value Theorem.

   (c) Use the Mean Value Theorem to show that

$$|\arctan x - \arctan y| \leq |x - y|, \qquad \forall x, y \in \mathbb{R}.$$

   (d) State Cauchy's Generalisation of the Mean Value Theorem.

4. (a) State and prove l'Hôpital's Rule for finding $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ when $f(c) = g(c) = 0$. Here you may use without proof Cauchy's Generalisation of the Mean Value Theorem.

   (b) Evaluate the following limits:

   (i) $\displaystyle \lim_{x \to -2} \frac{x^3 + 6x^2 + 12x + 8}{x^3 + 5x^2 + 8x + 4}$,

   (ii) $\displaystyle \lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^2 \tan^2 x$,

   (iii) $\displaystyle \lim_{x \to 0} \frac{\tan^3 x - \sin^3 x}{x^5}$.

5. In this question $I \subset \mathbb{R}$ is an interval. The endpoints of this interval may be real numbers or $\pm\infty$.

   (a) Define what it means for a function $F : I \to \mathbb{R}$ to be a *primitive* of the function $f : I \to \mathbb{R}$.

   (b) Suppose that the function $f : I \to \mathbb{R}$ has a primitive $F : I \to \mathbb{R}$. Is this primitive unique?

   (c) State and prove the Fundamental Theorem of Calculus.

   (d) Let $I \subset \mathbb{R}$ be an interval. Define what it means for a function $F : I \to \mathbb{R}$ to be an indefinite integral of the locally Riemann integrable function $f : I \to \mathbb{R}$.

   (e) State the theorem about the Properties of the Indefinite Integral.

   (f) Find the derivative of the function $G(x) = \displaystyle\int_{\cos x}^{\sin x} e^{t^2}\, dt$.

6. In this question $I \subset \mathbb{R}$ is an interval with endpoints c and d. These endpoints may be real numbers or $\pm\infty$.

   (a) Define what it means for a function $f : I \to \mathbb{R}$ to be locally Riemann integrable.

   (b) Let $f : I \to \mathbb{R}$ be locally Riemann integrable. Define what it means for f to be integrable over I in the improper sense.

   (c) State and prove the Comparison Theorem for Improper Integrals.

   You may use without proof the following fact. Let $f : I \to [0, +\infty)$ be a locally Riemann integrable function. Then f is integrable over I if and only if there exists a constant K such that for any $a, b \in I$, $a < b$, we have $\int_a^b f(x)\,dx \le K$.

   (d) Suppose that f is locally Riemann integrable over I and suppose that $\int_c^d |f(x)|\,dx$ exists. Prove that $\int_c^d f(x)\,dx$ exists.

   (e) Prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{1+x^4}\,dx$.

   [*Hint:* you may find it helpful to use the results from (c) and (d).]