MATH1102 Analysis

Solutions to examination paper for the academic year 2008/2009

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Solution to question 1

(a)(i) We say that $L = \lim\limits_{x\to x_0} f(x)$ if $\forall\varepsilon > 0 \;\; \exists\delta > 0$ such that

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon.$$

[3 marks; bookwork]

(a)(ii) We say that the function $f : D \to \mathbb{R}$ is *continuous* at the point x_0 if $\forall\varepsilon > 0 \;\; \exists\delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

[3 marks; bookwork]

(a)(iii) The function f is said to be *differentiable* at the point x_0 if

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In this case the limit is denoted by $f'(x_0)$ (or $\frac{df}{dx}(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$) and is called the *derivative* of f at x_0. **[3 marks; bookwork]**

(a)(iv) Let f be differentiable at the point x_0. Then we have

$$0 = f'(x_0) \cdot 0 = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0} (x - x_0) =$$
$$\lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right) = \lim_{x\to x_0} (f(x) - f(x_0)),$$

so

$$\lim_{x\to x_0} (f(x) - f(x_0)) = 0.$$

But we also have the trivial identity

$$\lim_{x\to x_0} f(x_0) = f(x_0).$$

Adding up the latter two identities we get $\lim\limits_{x\to x_0} f(x) = f(x_0)$, which means that the function is continuous at the point x_0. **[7 marks; bookwork]**

(b)(i) The ε, δ definition of a limit works here with $L = 0$ and $\delta = \varepsilon$. Hence, $\lim_{x\to 0} f(x) = 0$. **[3 marks; unseen exercise]**

(b)(ii) The function f is not continuous at the point 0 because $\lim_{x\to 0} f(x) = 0 \neq 1 = f(0)$. **[3 marks; unseen exercise]**

(b)(iii) The function f is not differentiable at the point 0 because it is not continuous at this point. **[3 marks; unseen exercise]**

Solution to question 2

(a) We say that the function $f : [a, b] \to \mathbb{R}$ achieves a *global maximum* at the point $c \in [a, b]$ if $f(x) \leq f(c)\,,\ \ \forall x \in [a, b]$. The number $f(c)$ is called the *maximum value* of f and is denoted $\max_{x\in[a,b]} f(x)$.

We say that the function $f : [a, b] \to \mathbb{R}$ achieves a *global minimum* at the point $d \in [a, b]$ if $f(x) \geq f(d)\,,\ \ \forall x \in [a, b]$. The number $f(d)$ is called the *minimum value* of f and is denoted $\min_{x\in[a,b]} f(x)$. **[3 marks; bookwork]**

(b) If $f : [a, b] \to \mathbb{R}$ is continuous then it achieves its global maximum at some $c \in [a, b]$ and its global minimum at some $d \in [a, b]$. **[4 marks; bookwork]**

(c) Suppose $f'(d) > 0$. Consider the function

$$g(x) := \begin{cases} \frac{f(x)-f(d)}{x-d} & \text{if} \quad x \neq d, \\ f'(d) & \text{if} \quad x = d. \end{cases}$$

This function is continuous at the point d, so by the Inertia Principle $\exists \delta > 0$ such that

$$|x - d| < \delta \quad \& \quad x \in [a, b] \qquad \Longrightarrow \qquad g(x) > 0.$$

Choose an arbitrary $\tilde{x} \in [a, b]$ such that $-\delta < \tilde{x} - d < 0$. Then

$$g(\tilde{x}) = \frac{f(\tilde{x}) - f(d)}{\tilde{x} - d} > 0$$

which implies $f(\tilde{x}) < f(d)$, contradicting the definition of a global minimum.

Similarly, the assumption $f'(d) < 0$ also leads to a contradiction.

This leaves us with $f'(d) = 0$ as the only possibility. **[9 marks; done in the lectures for the case of global maximum]**

(d)(i) The function $f : [0, 3] \to \mathbb{R}$, $f(x) = x^3 - 3x$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$ so the minimum is achieved either at the endpoints or at interior critical points. The only interior critical point is 1. Hence,

$$\min_{x\in[0,3]} (x^3 - 3x) = \min\{f(0), f(1), f(3)\} = \min\{0, -2, 18\} = -2.$$

[2 marks; unseen exercise]

(d)(ii) The function $f : [0, 3] \to \mathbb{R}$, $f(x) = 3x - x^3$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$ so the minimum is achieved either at the endpoints or at interior critical points. The only interior critical point is 1. Hence,

$$\min_{x\in[0,3]} (3x - x^3) = \min\{f(0), f(1), f(3)\} = \min\{0, 2, -18\} = -18.$$

[3 marks; unseen exercise]

(d)(iii) The function $f : [-3/2\,,\,3/2] \to \mathbb{R}$, $f(x) = 3|x| - x^3$ is continuous on $[-3/2\,,\,3/2]$ and differentiable on $(-3/2\,,\,3/2)\setminus\{0\}$ so the minimum is achieved either at the points $-3/2$, 0, $3/2$ or at critical points from $(-3/2\,,\,3/2)\setminus\{0\}$. To find the critical points we note that for $x \in (-3/2\,,0)$ we have $f(x) = -3x - x^3$ and for $x \in (0,\,3/2)$ we have $f(x) = 3x - x^3$. The only critical point in the set $(-3/2\,,\,3/2)\setminus\{0\}$ is 1. Hence,

$$\min_{x\in[-3/2\,,\,3/2]} (3|x| - x^3) = \min\{f(-3/2), f(0), f(1), f(3/2)\} = \min\{63/8, 0, 2, 9/8\} = 0.$$

[4 marks; unseen exercise]

Solution to question 3

(a) Put $h(x) := f(x) - g(x)$. Then $h'(x) = 0$ for all $x \in (-1, 1)$. I claim that

$$h(x) = h(0), \qquad \forall x \in (-1, 1). \tag{1}$$

Case 1: $x = 0$. Obviously, (1) is true.

Case 2: $x \in (0, 1)$. Applying the Mean Value Theorem to the function h on the interval $[0, x]$ we get

$$\frac{h(x) - h(0)}{x} = h'(c) \tag{2}$$

for some $c \in (0, x)$. But $h'(c) = 0$, so (2) implies (1).

Case 3: $x \in (-1, 0)$. Applying the Mean Value Theorem to the function h on the interval $[x, 0]$ we get

$$\frac{h(x) - h(0)}{x} = h'(c) \tag{3}$$

for some $c \in (x, 0)$. But $h'(c) = 0$, so (3) implies (1). **[7 marks; unseen exercise, but somewhat similar to a lemma proved in the lectures and an exercise from an exercise sheet]**

(b) In the lectures the radius of convergence was defined as follows. Introduce the set

$$S := \left\{ r \geq 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \;\text{ converges for } x = r \text{ or } x = -r \right\}. \tag{4}$$

Obviously, $\{0\} \subset S \subset [0, +\infty)$. The *radius of convergence* of a power series is the extended real number

$$R := \sup S$$

where S is the set (4).

Another possible definition is that the radius of convergence of a power series is the extended real number R such that for $|x| < R$ the series converges absolutely and for $|x| > R$ the series diverges. I will accept this definition even though it is not *a priori* obvious that R with these properties exists. **[3 marks; bookwork]**

(c) Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$.
[5 marks; bookwork]

(d) Fix an $x \neq 0$ and apply the ratio test:

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| = \left|\frac{(-1)^{n+2}\frac{1}{n+1}}{(-1)^{n+1}\frac{1}{n}}\right| |x| = \frac{|x|}{1+\frac{1}{n}} \to |x|$$

as $n \to \infty$. Hence, the power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is 1. [**4 marks; unseen exercise**]

(e) The results from (c) and (d) tell us that the function g is differentiable on $(-1, 1)$ and that its derivative is obtained by differentiating the power series for g term-by-term:

$$g'(x) = \left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}\right)' = \sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{x^n}{n}\right)' = \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1} = \sum_{k=0}^{\infty}(-x)^k = \frac{1}{1+x}$$

because we are looking at a geometric series. [**4 marks; unseen exercise**]

(f) Application of the result from (a) with $f(x) = \ln(1+x)$ and $g(x)$ as in (e) gives

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} + \text{const}, \qquad \forall x \in (-1, 1).$$

Setting $x = 0$ we conclude that const $= 0$. [**2 marks; unseen exercise**]

Solution to question 4

(a) Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that

$$(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)\,.$$

[**5 marks; bookwork**]

(b) Fix $x > 0$ and consider the function

$$g(t) := f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - \dots - \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x-t)^n\,, \quad (5)$$

where t is the independent variable. Since $f : (a, b) \to \mathbb{R}$ is $n+1$ times differentiable, $g : (a, b) \to \mathbb{R}$ is differentiable. Differentiating (5) we get

$$\begin{aligned} g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \dots \\ \dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] \\ = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n \end{aligned}$$

(all the terms cancelled out, apart from the last one). Applying Cauchy's Generalisation of the Mean Value Theorem to the functions $g(t)$ and $h(t) := (x-t)^{n+1}$ on the interval $[0, x]$ (which we can because g is continuous on $[0, x] \subset (a, b)$ and differentiable on $(0, x) \subset (a, b)$) we get

$$\frac{g(x) - g(0)}{h(x) - h(0)} = \frac{g'(\xi)}{h'(\xi)} = \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{-(n+1)(x-\xi)^n} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi \in (0, x)$. But

$$g(0) = R_n(x), \quad g(x) = 0, \quad h(0) = x^{n+1}, \quad h(x) = 0,$$

so the above formula can be rewritten as

$$\frac{-R_n(x)}{-x^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,,$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\, x^{n+1}$. **[11 marks; bookwork]**

(c) We have

$$\arctan' x = \frac{1}{1+x^2}\,, \qquad \arctan'' x = -\frac{2x}{(1+x^2)^2}\,,$$

so application of the result from (b) to the function $\arctan : \mathbb{R} \to \mathbb{R}$ with $n = 1$ gives

$$\arctan x = x - \frac{\xi}{(1+\xi^2)^2}\, x^2 \tag{6}$$

for some $\xi \in (0, x)$. **[5 marks; unseen exercise]**

(d) Take $x = \frac{1}{\sqrt{3}}$. Then formula (6) becomes

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{\xi}{3(1+\xi^2)^2}$$

for some $\xi \in (0, \frac{1}{\sqrt{3}})$. The function $\xi \mapsto \frac{\xi}{(1+\xi^2)^2}$ is strictly increasing on the interval $[0, \frac{1}{\sqrt{3}}]$, so

$$\frac{1}{\sqrt{3}} - \frac{\frac{1}{\sqrt{3}}}{3(1+\frac{1}{3})^2} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}\,,$$

or, equivalently, $2\sqrt{3} - \dfrac{3\sqrt{3}}{8} < \pi < 2\sqrt{3}$. **[4 marks; unseen exercise]**

Solution to question 5

(a) A *partition* P of the interval $[a, b]$ is a finite sequence of real numbers $P = \{x_0, x_1, \dots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

[1 mark; bookwork]

(b)(i) The *lower Darboux sum* of f with respect to the partition P is defined as

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1})\,,$$

where $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$. The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})\,,$$

where $M_i := \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$. **[3 marks; bookwork]**

(b)(ii) Write the original partition P as

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the partition P' as

$$a = x_0 < x_1 < \cdots < x_{k-1} < x' < x_k < \cdots < x_{n-1} < x_n = b$$

Then

$$\begin{aligned} L(f,P') - L(f,P) = \left(\inf_{x\in[x_{k-1},x']} f(x)\right)(x' - x_{k-1}) + \left(\inf_{x\in[x',x_k]} f(x)\right)(x_k - x') \\ - \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x_k - x_{k-1}). \end{aligned}$$

But

$$\inf_{x\in[x_{k-1},x']} f(x) \geq \inf_{x\in[x_{k-1},x_k]} f(x), \qquad \inf_{x\in[x',x_k]} f(x) \geq \inf_{x\in[x_{k-1},x_k]} f(x),$$

so

$$\begin{aligned} L(f,P') - L(f,P) \geq \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x' - x_{k-1}) + \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x_k - x') \\ - \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x_k - x_{k-1}) = 0. \end{aligned}$$

The inequality $U(f,P') \leq U(f,P)$ is proved in a similar way. **[6 marks; bookwork]**

(b)(iii) Repeated application of the result from (b)(ii). **[1 mark; bookwork]**

(b)(iv) Let $R := P \cup Q$ be a common refinement of P and Q. Then, by the result from (b)(iii),

$$L(f,P) \leq L(f,R), \qquad U(f,R) \leq U(f,Q).$$

But, obviously, $L(f,R) \leq U(f,R)$, so

$$L(f,P) \leq L(f,R) \leq U(f,R) \leq U(f,Q).$$

[3 marks; bookwork]

(b)(v) The *lower Riemann integral* of f over $[a,b]$ is defined as

$$\underline{\int_a^b} f(x)\,dx := \sup_{P\in\mathcal{P}[a,b]} L(f,P)\,.$$

The *upper Riemann integral* of f over $[a,b]$ is defined as

$$\overline{\int_a^b} f(x)\,dx := \inf_{P\in\mathcal{P}[a,b]} U(f,P)\,.$$

[3 marks; bookwork]

(b)(vi) Immediate consequence of the result from (b)(iv) and the definition from (b)(v). **[2 marks; bookwork]**

(b)(vii) The function f is said to be *Riemann integrable* on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx\,.$$

In this case the common value of the lower and upper Riemann integrals is denoted by

$$\int_a^b f(x)\,dx$$

and is called the *Riemann integral* of f over $[a,b]$. **[2 marks; bookwork]**

(c) Consider the function $f : [0,1] \to \mathbb{R}$ defined as follows:

$$f(x) := \begin{cases} 0 & \text{if} \quad x < 1/2, \\ 1 & \text{if} \quad x \geq 1/2. \end{cases}$$

This function is Riemann integrable on $[0,1]$ because it is monotone (increasing) but it is not continuous because $\lim\limits_{x \to 1/2-} f(x) \neq f(1/2)$. **[4 marks; unseen exercise]**

Solution to question 6

(a)(i)

$$\begin{aligned} T_n - T_{n+1} = \int_n^{n+1} f(x)\,dx - f(n+1) = \int_n^{n+1} f(x)\,dx - \int_n^{n+1} f(n+1)\,dx \\ = \int_n^{n+1} (f(x) - f(n+1))\,dx \geq 0 \end{aligned}$$

so the sequence $\{T_n\}$ is decreasing. **[4 marks; bookwork]**

(a)(ii)

$$T_n = \sum_{k=1}^{n-1} \int_k^{k+1} f(k)\,dx + f(n) - \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,dx = \sum_{k=1}^{n-1} \int_k^{k+1} (f(k) - f(x))\,dx + f(n) \geq f(n).$$

[8 marks; bookwork]

(a)(iii) The estimate from (a)(ii) and the fact that f is nonnegative imply that

$$T_n \geq 0.$$

We are looking at a decreasing sequence $\{T_n\}$ which is bounded from below. Such a sequence has to converge. **[3 marks; bookwork]**

(a)(iv) The definition of T_n and the result from (a)(iii) imply that the series $\sum\limits_{k=1}^{\infty} f(k)$ converges iff $\lim_{n\to\infty} \int\limits_1^n f(x)\,dx$ exists. Observe now that for any $b \in \mathbb{R}$, $b \geq 1$, we have

$$\int_1^{[b]} f(x)\,dx \leq \int_1^b f(x)\,dx \leq \int_1^{[b]+1} f(x)\,dx\,,$$

where $[b]$ denotes the integer part of the real number b (greatest integer less than or equal to b). Hence, $\lim_{n\to\infty} \int\limits_1^n f(x)\,dx$ exists if and only if $\lim_{b\to+\infty} \int\limits_1^b f(x)\,dx$ exists. In other words, $\lim_{n\to\infty} \int\limits_1^n f(x)\,dx$ exists if and only if the improper integral $\int\limits_1^\infty f(x)\,dx$ exists (converges). **[3 marks; bookwork]**

(b) Firstly, rewrite our series as

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}\,.$$

Introduce the function $f : [1, \infty) \to [0, \infty)$, $f(x) := \dfrac{1}{(x+1)\ln(x+1)}$. This function is decreasing so by (a)(iv) the above series converges if and only if

$$\int_1^{\infty} \frac{dx}{(x+1)\ln(x+1)}$$

converges. But the divergence of the latter integral is established by explicit calculations:

$$\int_1^{b} \frac{dx}{(x+1)\ln(x+1)} = \ln(\ln(x+1))\big|_{x=1}^{x=b} = \ln(\ln(b+1)) - \ln(\ln 2) \to +\infty \qquad \text{as} \qquad b \to +\infty.$$

[7 marks; unseen exercise]