All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$.
   (i) Define, in terms of ε and δ, the meaning of the mathematical statement "the function f has limit L at the point x_0".
   (ii) Define, in terms of ε and δ, the meaning of the mathematical statement "the function f is continuous at the point x_0".
   (iii) Define the meaning of the mathematical statement "the function f is differentiable at the point x_0".
   (iv) Prove that if f is differentiable at x_0, then f is continuous at x_0.

   (b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows: $f(x) = \begin{cases} x & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0. \end{cases}$
   (i) Does $\lim_{x \to 0} f(x)$ exist? Justify your answer. If $\lim_{x \to 0} f(x)$ exists, evaluate it.
   (ii) Is f continuous at the point 0? Justify your answer.
   (iii) Is f differentiable at the point 0? Justify your answer. If f is differentiable at the point 0, evaluate $f'(0)$.

2. (a) Define what it means for a function $f : [a, b] \to \mathbb{R}$ to achieve a *global maximum* at a point $c \in [a, b]$ and a *global minimum* at a point $d \in [a, b]$.
   (b) State the attainment of bounds theorem.
   (c) Suppose that the function $f : [a, b] \to \mathbb{R}$ is differentiable at the point $d \in (a, b)$ and suppose that f achieves a global minimum at the point d. Prove that $f'(d) = 0$.
   (d) Find, with justification,
   (i) $\min_{x \in [0,3]} (x^3 - 3x)$,
   (ii) $\min_{x \in [0,3]} (3x - x^3)$,
   (iii) $\min_{x \in [-3/2\,,\,3/2]} (3|x| - x^3)$.

3. (a) Suppose that the functions $f, g : (-1,1) \to \mathbb{R}$ are differentiable and that $f'(x) = g'(x)$ for all $x \in (-1,1)$. Prove that $f(x) = g(x) + \text{const}$ for all $x \in (-1,1)$.

(b) Define the notion of the radius of convergence of a power series.

(c) State the theorem about the differentiability of power series.

(d) Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$.

(e) Consider the function $g : (-1,1) \to \mathbb{R}$, $g(x) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$. Prove that $g'(x) = \dfrac{1}{1+x}$.

(f) Prove that $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ for all $x \in (-1,1)$.

Hint: you may find it helpful to use the results from (a) and (e).

4. (a) State Cauchy's generalisation of the mean value theorem.

(b) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a,b) \to \mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

Given an $x \in (0,b)$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$ for some $\xi \in (0,x)$.

(c) Let $x > 0$. Prove that $\arctan x = x - \dfrac{\xi}{(1+\xi^2)^2}\,x^2$ for some $\xi \in (0,x)$.

(d) Prove that $2\sqrt{3} - \dfrac{3\sqrt{3}}{8} < \pi < 2\sqrt{3}$.

5. (a) What is a partition of an interval $[a,b]$?

   (b) Let $f : [a,b] \to \mathbb{R}$ be bounded.

   (i) Define the lower $L(f,P)$ and upper $U(f,P)$ Darboux sums of f with respect to a given partition P of the interval $[a,b]$.

   (ii) Let the partition P' be a refinement of the partition P with one extra point. Prove that $L(f,P) \le L(f,P')$ and $U(f,P') \le U(f,P)$.

   (iii) Let the partition P' be a refinement of the partition P. Prove that $L(f,P) \le L(f,P')$ and $U(f,P') \le U(f,P)$.

   (iv) Prove that $L(f,P) \le U(f,Q)$ for any pair of partitions P and Q.

   (v) Define the lower Riemann integral $\underline{\int_a^b} f(x)\,dx$ and the upper Riemann integral $\overline{\int_a^b} f(x)\,dx$.

   (vi) Prove that $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$.

   (vii) Define what it means for f to be Riemann integrable on $[a,b]$.

   (c) Give, with justification, an example of a bounded function $f : [0,1] \to \mathbb{R}$ which is not continuous on $[0,1]$ but is Riemann integrable on $[0,1]$. Here you may use all the theorems and lemmata from the lecture course.

6. (a) Let $f : [1,\infty) \to [0,\infty)$ be a decreasing function and let

$$T_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx\,, \qquad n \in \mathbb{N}.$$

   (i) Prove that the sequence $\{T_n\}$ is decreasing.

   (ii) Prove that $T_n \ge f(n)$, $n \in \mathbb{N}$.

   (iii) Prove that the sequence $\{T_n\}$ converges.

   (iv) Prove that the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ exists (converges).

   (b) Prove that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.