MATH1102 Analysis

Solutions to examination paper for the academic year 2007/2008

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February 15, 2012

Solution to question 1

(a) The function f is said to be *differentiable* at the point x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In this case the limit is denoted by $f'(x_0)$ (or $\frac{df}{dx}(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$) and is called the *derivative* of f at x_0. **[4 marks; bookwork]**

(b) Take an arbitrary $x_0 > 0$. Then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0+h)^{\frac{1}{q}} - (x_0)^{\frac{1}{q}}}{h}$$

$$= \lim_{h \to 0} \frac{((x_0+h)^{\frac{1}{q}} - (x_0)^{\frac{1}{q}})((x_0+h)^{\frac{q-1}{q}} + (x_0+h)^{\frac{q-2}{q}}(x_0)^{\frac{1}{q}} + \ldots + (x_0+h)^{\frac{1}{q}}(x_0)^{\frac{q-2}{q}} + (x_0)^{\frac{q-1}{q}})}{h((x_0+h)^{\frac{q-1}{q}} + (x_0+h)^{\frac{q-2}{q}}(x_0)^{\frac{1}{q}} + \ldots + (x_0+h)^{\frac{1}{q}}(x_0)^{\frac{q-2}{q}} + (x_0)^{\frac{q-1}{q}})}$$

$$= \lim_{h \to 0} \frac{\left((x_0+h)^{\frac{1}{q}}\right)^q - \left((x_0)^{\frac{1}{q}}\right)^q}{h((x_0+h)^{\frac{q-1}{q}} + (x_0+h)^{\frac{q-2}{q}}(x_0)^{\frac{1}{q}} + \ldots + (x_0+h)^{\frac{1}{q}}(x_0)^{\frac{q-2}{q}} + (x_0)^{\frac{q-1}{q}})}$$

$$= \lim_{h \to 0} \frac{1}{(x_0+h)^{\frac{q-1}{q}} + (x_0+h)^{\frac{q-2}{q}}(x_0)^{\frac{1}{q}} + \ldots + (x_0+h)^{\frac{1}{q}}(x_0)^{\frac{q-2}{q}} + (x_0)^{\frac{q-1}{q}}}$$

$$\overset{\text{by the Algebra of Limits and continuity of } x^\alpha}{=} \frac{1}{\underbrace{(x_0)^{\frac{q-1}{q}} + (x_0)^{\frac{q-1}{q}} + \ldots + (x_0)^{\frac{q-1}{q}} + (x_0)^{\frac{q-1}{q}}}_{q \text{ terms}}}$$

$$= \frac{1}{q(x_0)^{\frac{q-1}{q}}} = \frac{1}{q}(x_0)^{\frac{1}{q}-1}.$$

[6 marks; question from an exercise sheet]

(c)

Theorem 1 *If f is differentiable at the point x_0, then f is continuous at the point x_0.*

[3 marks; bookwork]

(d)

Theorem 2 *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable at the point $x_0 \in (a, b)$. Then fg is differentiable at x_0 and*

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0). \tag{1}$$

[3 marks; bookwork]

Proof of Theorem 2 Let $h \neq 0$ be any number small enough so that $x_0 + h \in (a, b)$. We have

$$\begin{aligned}
&\frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&\quad = \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0+h) + f(x_0)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&\quad = \frac{f(x_0+h) - f(x_0)}{h}\, g(x_0+h) + f(x_0)\, \frac{g(x_0+h) - g(x_0)}{h}\ ,
\end{aligned}$$

so by the Algebra of Limits and Theorem 1

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&\quad = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}\ \lim_{h\to 0} g(x_0+h) + f(x_0)\ \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\
&\quad = f'(x_0)\, g(x_0)\ +\ f(x_0)\, g'(x_0)\, ,
\end{aligned}$$

which means that the function fg is differentiable at x_0 and its derivative is given by formula (1). □ **[6 marks; bookwork]**

(e)

$$(f_1 f_2 f_3 \ldots f_{p-1} f_p)' = f_1' f_2 f_3 \ldots f_{p-1} f_p + f_1 f_2' f_3 \ldots f_{p-1} f_p + \ldots f_1 f_2 f_3 \ldots f_{p-1} f_p'\, .$$

[1 mark; unseen exercise]

(f)

$$(x^{p/q})' = ((x^{1/q})^p)' = p(x^{1/q})^{p-1}(x^{1/q})' = p(x^{1/q})^{p-1}\frac{1}{q}x^{1/q-1} = \frac{p}{q}x^{p/q-1}.$$

[2 marks; question from an exercise sheet, however there students were asked to do the question differently, without recourse to the Product Rule]

Solution to question 2

(a)(i) Consider two power series:

$$\sum_{n=0}^{\infty} a_n x^n \tag{2}$$

and

$$\sum_{n=1}^{\infty} n a_n x^n \tag{3}$$

Denote the radii of convergence of the power series (2) and (3) by R and R' respectively.

Lemma 1 *The two power series (2) and (3) have the same radii of convergence.*

Proof Let us prove first that

$$R' \leq R. \tag{4}$$

Suppose (4) is false. Then $R' > R$ and we can choose an x such that $R < x < R'$. Then the series (3) converges absolutely whereas the series (2) diverges. But

$$|a_n x^n| \leq |n a_n x^n|$$

for $n \geq 1$ so the series (2) converges by the comparison test. Thus, the series (2) diverges and converges, which is impossible. This contradiction proves (4).

Let us prove now that

$$R \leq R'. \tag{5}$$

Suppose (5) is false. Then $R > R'$ and we can choose y, z such that $R' < y < z < R$. Then the series

$$\sum_{n=0}^{\infty} a_n z^n \tag{6}$$

converges absolutely whereas the series

$$\sum_{n=0}^{\infty} n a_n y^n \tag{7}$$

diverges. But

$$|n a_n y^n| = |a_n z^n| \left(n \left| \frac{y}{z} \right|^n \right). \tag{8}$$

It is known (MATH1101) that $\lim\limits_{n\to\infty} \left(n\left|\frac{y}{z}\right|^n\right) = 0$ so there exists an $M \geq 0$ such that $n\left|\frac{y}{z}\right|^n \leq M$, $\forall n \in \mathbb{N}$, and formula (8) implies $|n a_n y^n| \leq M|a_n z^n|$. Hence the series (7) converges by the comparison test. Thus, the series (7) diverges and converges, which is impossible. This contradiction proves (5).

Formulae (4) and (5) imply $R = R'$. □ **[9 marks; bookwork (special case of a lemma proved in lectures)]**

(a)(ii) Now consider the power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1}. \tag{9}$$

Note that (9) is the formal derivative of the original power series (2).

Lemma 2 *The two power series (2) and (9) have the same radii of convergence.*

Proof The two power series obviously converge for $x = 0$. For $x \neq 0$ we have

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} n a_n x^n$$

and the required result follows from Lemma 1. □ **[3 marks; bookwork]**

(a)(iii) Fix an $x_0 \in (-R, R)$ and an h_0 such that $0 < h_0 < R - |x_0|$, and let h be such that $0 < |h| < h_0$. Then by Lemma 2 and the inequality stated in the examination paper

$$\left| \frac{\sum\limits_{n=0}^{\infty} a_n (x_0+h)^n - \sum\limits_{n=0}^{\infty} a_n x_0^n}{h} - \sum_{n=1}^{\infty} n a_n x_0^{n-1} \right| \leq \frac{|h|}{2} \underbrace{\sum_{n=2}^{\infty} n(n-1)|a_n|(|x_0| + h_0)^{n-2}}_{\text{number, not a function of } h} \to 0$$

as $h \to 0$. $\square$ **[6 marks; bookwork]**

(b)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

[3 marks; example done in lectures]

(c) By (b) and (a)(iii) we have

$$\left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} n x^{n-1}.$$

But

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

so

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

[4 marks; unseen exercise]

Solution to question 3

(a) We know the following fact which we are allowed to use without proof.

Lemma 3 *Let D be an interval and x_0 be an interior point of D. Suppose that the function $f : D \to \mathbb{R}$ is differentiable at the point x_0, and suppose that f achieves a global maximum (minimum) at the point x_0. Then $f'(x_0) = 0$.*

The following theorem is known as *Rolle's Theorem.*

Theorem 3 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and let $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.*

[4 marks; bookwork]
Proof According to the Attainment of Bounds Theorem there exist $x_0, x_0' \in [a, b]$ such that

$$f(x_0') \le f(x) \le f(x_0), \qquad \forall x \in [a, b].$$

If both x_0 and x_0' are endpoints of our interval then $f(x) = \text{const}$ and $f'(x) = 0$, $\forall x \in (a, b)$, so any $c \in (a, b)$ would do. If x_0 is not an endpoint then, by Lemma 3, $f'(x_0) = 0$, so we can take $c = x_0$. If x_0' is not an endpoint then, by Lemma 3, $f'(x_0') = 0$, so we can take $c = x_0'$. $\square$ **[5 marks; bookwork]**

(b) The following theorem is known as *Mean Value Theorem*, or simply as *MVT*. It is also known as *Lagrange's Theorem.*

Theorem 4 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{10}$$

[4 marks; bookwork]

Proof Let us introduce a new function $g(x) = f(x) - kx$ where the constant k is chosen from the condition that the function g satisfies the conditions of Rolle's Theorem:

$$g(a) = g(b) \quad \Longleftrightarrow \quad f(a) - ka = f(b) - kb \quad \Longleftrightarrow \quad k = \frac{f(b) - f(a)}{b - a}\,.$$

Thus, the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}\,x$$

satisfies the conditions of Rolle's Theorem. Rolle's Theorem tell us that there exists a point $c \in (a, b)$ such that

$$g'(c) = 0 \quad \Longleftrightarrow \quad f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \quad \Longleftrightarrow \quad f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

$\square$ **[5 marks; bookwork]**

(c) Applying the Mean Value Theorem to the function sin on the interval $[0, \frac{\pi}{6}]$ we conclude that

$$\frac{\sin\frac{\pi}{6} - \sin 0}{\frac{\pi}{6} - 0} = \cos c \tag{11}$$

for some

$$c \in \left(0, \frac{\pi}{6}\right) \tag{12}$$

But $\sin\frac{\pi}{6} = \frac{1}{2}$ so formula (11) implies

$$\pi = \frac{3}{\cos c}\,. \tag{13}$$

Formula (12) give us the inequalities

$$\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} < \cos c < \cos 0 = 1. \tag{14}$$

Combining (13) with (14) we get $3 < \pi < 2\sqrt{3}$. **[7 marks; unseen exercise]**

Solution to question 4

(a) The following theorem is known as *L'Hôpital's Rule.*

Theorem 5 *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable, and let $c \in (a, b)$ be such that $f(c) = g(c) = 0$, $g'(x) \neq 0$ for $x \neq c$. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{15}$$

provided the latter limit exists.

[4 marks; bookwork]

Note that $g(x) \neq 0$ for $x \neq c$ (via Rolle's Theorem).

Proof of Theorem 5 Suppose $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists and equals L. Then

$$\lim_{x\to c+} \frac{f'(x)}{g'(x)} = \lim_{x\to c-} \frac{f'(x)}{g'(x)} = L.$$

In order to prove (15) we need to show that

$$\lim_{x\to c+} \frac{f(x)}{g(x)} = L \tag{16}$$

and

$$\lim_{x\to c-} \frac{f(x)}{g(x)} = L. \tag{17}$$

Let us prove (16). Take an arbitrary sequence $\{x_n\} \in (c, b)$ such that

$$\lim_{n\to\infty} x_n = c. \tag{18}$$

Applying Cauchy's Generalisation of the Mean Value Theorem on the interval $[c, x_n]$ we get a sequence $\{\xi_n\}$ such that

$$c < \xi_n < x_n \tag{19}$$

and

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(\xi_n)}{g'(\xi_n)} . \tag{20}$$

Formulae (18), (19) and the Sandwich Principle imply

$$\lim_{n\to\infty} \xi_n = c. \tag{21}$$

Formula (21) and the sequential definition of a limit (one-sided version) imply

$$\lim_{n\to\infty} \frac{f'(\xi_n)}{g'(\xi_n)} = L. \tag{22}$$

Formulae (20) and (22) imply

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L. \tag{23}$$

Formula (23) and the sequential definition of a limit (one-sided version) imply (16).

Formula (17) is proved similarly. □ **[7 marks; bookwork]**

(b)(i)

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{\cos 0}{6} = \frac{1}{6} .$$

[3 marks; unseen exercise]

(b)(ii)

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = \lim_{x\to 0} \frac{1}{1+x} = \frac{1}{1+0} = 1.$$

[2 mark; unseen exercise]

(b)(iii) Note that for $x \neq 0$ we have

$$(1+x)^{1/x} = e^{\frac{\ln(1+x)}{x}} .$$

Introduce the function

$$f(x) = \begin{cases} \frac{\ln(1+x)}{x} & \text{for} \quad x \neq 0, \\ 1 & \text{for} \quad x = 0. \end{cases}$$

This function is defined in a neighbourhood of 0 and by (ii)(b) is continuous at 0. The function $e^{f(x)}$ is also continuous at 0 as a composition of continuous functions. Hence

$$\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} e^{f(x)} = e^{f(0)} = e.$$

[4 marks; unseen exercise]

(b)(iv)

$$\lim_{x\to 0}\frac{\sin^2 x-\tan^2 x}{x^4} = -\lim_{x\to 0}\frac{\sin^4 x}{x^4\cos^2 x} = -\left(\lim_{x\to 0}\frac{1}{\cos^2 x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^4$$
$$= -\left(\lim_{x\to 0}\frac{1}{\cos^2 x}\right)\left(\lim_{x\to 0}\cos x\right)^4 = -\frac{1}{\cos^2 0}\cos^4 0 = -1.$$

[5 marks; unseen exercise]

Solution to question 5

(a) A *partition* P of the interval $[a,b]$ is a finite sequence of real numbers $P=\{x_0,x_1,\dots,x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

[2 marks; bookwork]

(b)(i) The *lower Darboux sum* of f with respect to the partition P is defined as

$$L(f,P) := \sum_{i=1}^{n} m_i(x_i - x_{i-1})\,,$$

where $m_i := \inf_{x\in[x_{i-1},x_i]} f(x)$. The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f,P) := \sum_{i=1}^{n} M_i(x_i - x_{i-1})\,,$$

where $M_i := \sup_{x\in[x_{i-1},x_i]} f(x)$. **[3 marks; bookwork]**

(b)(ii) The *lower Riemann integral* of f over $[a,b]$ is defined as

$$\underline{\int_a^b} f(x)\,dx := \sup_{P\in\mathcal{P}[a,b]} L(f,P)\,.$$

The *upper Riemann integral* of f over $[a,b]$ is defined as

$$\overline{\int_a^b} f(x)\,dx := \inf_{P\in\mathcal{P}[a,b]} U(f,P)\,.$$

[3 marks; bookwork]

(b)(iii) The function f is said to be *Riemann integrable* on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx\,.$$

In this case the common value of the lower and upper Riemann integrals is denoted by

$$\int_a^b f(x)\,dx$$

and is called the *Riemann integral* of f over $[a,b]$. **[2 marks; bookwork]**

(b)(iv)

Theorem 6 *Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ iff $\forall \varepsilon > 0 \;\; \exists P \in \mathcal{P}[a,b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{24}$$

[3 marks; bookwork]

(c) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if} \quad x \quad \text{is rational,} \\ 0 & \text{if} \quad x \quad \text{is irrational.} \end{cases}$

Let $P = \{x_0, \dots, x_n\}$ be any partition of $[0,1]$. Any interval $[x_{i-1}, x_i]$ contains both rationals and irrationals, therefore

$$\inf_{x \in [x_{i-1}, x_i]} f(x) = 0, \qquad \sup_{x \in [x_{i-1}, x_i]} f(x) = 1.$$

Multiplying by $(x_i - x_{i-1})$ and summing over i we get

$$L(f,P) = 0, \qquad U(f,P) = 1.$$

Thus,

$$\underline{\int_0^1} f(x)\,dx := \sup_{P \in \mathcal{P}[0,1]} L(f,P) = 0,$$
$$\overline{\int_0^1} f(x)\,dx := \inf_{P \in \mathcal{P}[0,1]} U(f,P) = 1,$$

and as the two are different the function f is not Riemann integrable. **[4 marks; question from an exercise sheet]**

(d) A function $f : [a,b] \to \mathbb{R}$ is said to be *decreasing* on $[a,b]$ if

$$x_1 < x_2 \quad \& \quad x_1, x_2 \in [a,b] \qquad \Longrightarrow \qquad f(x_1) \geq f(x_2).$$

[2 marks; bookwork]

(e) Proof Consider a partition P_n into n subintervals of equal length: $P_n = \{x_0, x_1, \dots, x_n\}$,

$$x_i = a + \frac{b-a}{n} i, \qquad i = 0, 1, \dots, n.$$

Then

$$L(f,P_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i), \qquad U(f,P_n) = \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i),$$
$$U(f,P_n) - L(f,P_n) = \frac{b-a}{n}(f(x_0) - f(x_n)) = \frac{b-a}{n}(f(a) - f(b)).$$

Now, let ε be an arbitrary positive number. Choose n so large that

$$\frac{b-a}{n}(f(a) - f(b)) < \varepsilon.$$

Then

$$U(f,P_n) - L(f,P_n) < \varepsilon$$

and Riemann's Criterion for Integrability tells us that f is Riemann integrable on $[a,b]$. □ **[6 marks; for the case of increasing function proof done in lectures]**

Solution to question 6

(a) A *primitive* of $f : [a,b] \to \mathbb{R}$ is a function $F : [a,b] \to \mathbb{R}$ which is continuous on $[a,b]$, differentiable on (a,b) and satisfies $F'(x) = f(x)$ on (a,b). **[2 marks; bookwork]**

(b) No: one can always add a constant to F. **[1 mark; bookwork]**

(c) The following theorem is known as the *Fundamental Theorem of Calculus.*

Theorem 7 *Let f be Riemann integrable on $[a,b]$ and let F be a primitive of f. Then*

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a) \;=\; F(x)|_a^b\,.$$

[4 marks; bookwork]

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Applying the Mean Value Theorem to the function F on the subinterval $[x_{i-1}, x_i]$ we get

$$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1})$$

for some $t_i \in (x_{i-1}, x_i)$. But

$$m_i \le f(t_i) \le M_i$$

where I use the standard notation $m_i := \inf\limits_{x\in[x_{i-1},x_i]} f(x)$, $M_i := \sup\limits_{x\in[x_{i-1},x_i]} f(x)$. Hence,

$$m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1}).$$

Summing up over i from 1 to n we get

$$L(f,P) \le \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \le U(f,P).$$

But

$$\sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(x_n) - F(x_0) = F(b) - F(a),$$

so the above inequality can be rewritten as

$$L(f,P) \le F(b) - F(a) \le U(f,P). \tag{25}$$

The inequality (25) is true for any partition P. Taking a limiting sequence of partitions $\{P_n\}$ and letting $n \to \infty$ we get from (25)

$$\lim_{n\to\infty} L(f,P_n) \le F(b) - F(a) \le \lim_{n\to\infty} U(f,P_n).$$

It remains to note that as f Riemann integrable on $[a,b]$ the Theorem about limiting sequences tells us that

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n) = \int_a^b f(x)\,dx\,.$$

□ **[8 marks; bookwork]**

(d) An *indefinite integral* of f is a function $F : [a,b] \to \mathbb{R}$ defined by

$$F(x) = \int_c^x f(t)\,dt$$

for some $c \in [a,b]$. **[2 marks; bookwork]**

(e)

Theorem 8 *Let $f \in \mathcal{R}_{loc}(I)$ and let $F : I \to \mathbb{R}$ be an indefinite integral of f. Then*

(i) F is continuous on I;

(ii) F is differentiable at each interior point $x_0 \in I$ at which f is continuous, and at such a point $F'(x_0) = f(x_0)$;

(iii) if f is continuous on I then F is a primitive of f.

[4 marks; bookwork]

(f) We have

$$G(x) = F(\cos x) - F(\sin x)$$

where

$$F(y) = \int_0^y e^t \, dt \, .$$

Hence, by the Chain Rule and the Theorem about the Properties of the Indefinite Integral we have

$$G'(x) = -F'(\cos x) \sin x - F'(\sin x) \cos x = -e^{\cos x} \sin x - e^{\sin x} \cos x.$$

[4 marks; unseen exercise]