*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) State the definition of the derivative of a function $f : (0, +\infty) \to \mathbb{R}$ at the point $x_0 \in (0, +\infty)$.

   (b) Consider the function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = x^{1/q}$, where $q \in \mathbb{N}$. Using the definition of the derivative prove that $f'(x) = \dfrac{1}{q}x^{1/q-1}$.

   *Hint:* use the formula for $a^n - b^n$ with $a = (x_0 + h)^{1/q}$, $b = (x_0)^{1/q}$, $n = q$.

   (c) State the theorem about the relationship between differentiability and continuity.

   (d) State and prove the Product Rule.

   (e) How would the Product Rule look for a product of $p$ functions $f_1, f_2, \ldots f_p$?

   (f) Consider the function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = x^{p/q}$, where $p, q \in \mathbb{N}$. Prove that $f'(x) = \dfrac{p}{q}x^{p/q-1}$.

   *Hint:* you may find it helpful to use the results from (e) and (b).

2. (a) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.

   (i) Prove that the power series $\sum_{n=1}^{\infty} n a_n x^n$ has radius of convergence $R$.

   (ii) Prove that the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence $R$.

   (iii) Consider the function $f : (-R, R) \to \mathbb{R}$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Prove that $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Here you may use without proof the fact that

   $$\left| \frac{(x_0 + h)^n - x_0^n}{h} - n x_0^{n-1} \right| \leq \frac{n(n-1)|h|}{2}(|x_0| + |h|)^{n-2},$$

   $\forall x_0, h \in \mathbb{R}$, $h \neq 0$, $\forall n \in \mathbb{N}$, $n \geq 2$.

   (b) Write down the power series for the function $f : (-1, 1) \to \mathbb{R}$, $f(x) = \dfrac{1}{1-x}$.

   (c) Write down the power series for the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = \dfrac{1}{(1-x)^2}$.

   *Hint:* you may find it helpful to use the results from (b) and (a)(iii).

3. (a) State and prove Rolle's Theorem. Here you may use without proof the following fact: if the function $f : [a, b] \to \mathbb{R}$ achieves a global maximum (minimum) at the point $x_0 \in (a, b)$ and if $f$ is differentiable at the point $x_0$ then $f'(x_0) = 0$.

   (b) State and prove the Mean Value Theorem.

   (c) Prove that $3 < \pi < 2\sqrt{3}$.

   *Hint:* apply the Mean Value Theorem to the function sin on the interval $[0, \frac{\pi}{6}]$.

4. (a) State and prove l'Hôpital's Rule for finding $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ when $f(c) = g(c) = 0$. Here you may use without proof Cauchy's Generalisation of the Mean Value Theorem.

   (b) Evaluate the following limits:

   (i) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$ ,

   (ii) $\displaystyle\lim_{x\to 0} \frac{\ln(1 + x)}{x}$ ,

   (iii) $\displaystyle\lim_{x\to 0} (1 + x)^{1/x}$ ,

   (iv) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x - \tan^2 x}{x^4}$ .

5. (a) What is a partition of an interval $[a, b]$?

   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded.

   (i) Define the lower and upper Darboux sums of $f$ with respect to a given partition of the interval $[a, b]$

   (ii) Define the lower and upper Riemann integrals of $f$ over $[a, b]$.

   (iii) Define what it means for $f$ to be Riemann integrable on $[a, b]$.

   (iv) State Riemann's Criterion for Integrability.

   (c) Give, with justification, an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable on $[0, 1]$.

   (d) Define what it means for a function $f : [a, b] \to \mathbb{R}$ to be decreasing.

   (e) Prove that a decreasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. Here you may use without proof Riemann's Criterion for Integrability.

6. (a) Define what it means for a function $F : [a, b] \to \mathbb{R}$ to be a primitive of the function $f : [a, b] \to \mathbb{R}$.

   (b) Suppose that the function $f : [a, b] \to \mathbb{R}$ has a primitive $F : [a, b] \to \mathbb{R}$. Is this primitive unique?

   (c) State and prove the Fundamental Theorem of Calculus.

   (d) Define what it means for a function $F : [a, b] \to \mathbb{R}$ to be an indefinite integral of the Riemann integrable function $f : [a, b] \to \mathbb{R}$.

   (e) State the Theorem about the properties of the indefinite integral.

   (f) Find the derivative of the function $G(x) = \displaystyle\int_{\sin x}^{\cos x} e^t \, dt$.