MATH1102 Analysis

Solutions to examination paper for the academic year 2006/2007

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Solution to question 1

(i)(a). We say that the function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at the point x_0 if $\forall \varepsilon > 0 \ \ \exists \delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

[5 marks; bookwork]

(i)(b). The function f is said to be *differentiable* at the point x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In this case the limit is denoted by $f'(x_0)$ (or $\frac{df}{dx}(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$) and is called the *derivative* of f at x_0. **[5 marks; bookwork]**

(i)(c). Let f be differentiable at the point x_0. Then we have

$$0 = f'(x_0) \cdot 0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) =$$
$$\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right) = \lim_{x \to x_0} (f(x) - f(x_0)),$$

so

$$\lim_{x \to x_0} (f(x) - f(x_0)) = 0.$$

But we also have the trivial identity

$$\lim_{x \to x_0} f(x_0) = f(x_0).$$

Adding up the latter two identities we get $\lim\limits_{x \to x_0} f(x) = f(x_0)$, which means that the function is continuous at the point x_0. **[7 marks; bookwork]**

(ii). Take an arbitrary $x_0 \in \mathbb{R}$. Then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0 + h)^n - (x_0)^n}{h} =$$
$$\lim_{h \to 0} \big(\underbrace{(x_0 + h)^{n-1} + (x_0 + h)^{n-2} x_0 + \ldots + (x_0 + h) x_0^{n-2} + x_0^{n-1}}_{n \text{ terms}} \big) = n x_0^{n-1}$$

by the continuity of polynomials. **[5 marks; example done in lectures]**

(iii). Put $x_n = \dfrac{1}{\frac{\pi}{2} + 2\pi n}$, $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} x_n = 0$$

but

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1 \neq 0 = f(0),$$

so f is discontinuous at 0. By (i)(c) f is not differentiable at 0. **[3 marks; unseen exercise]**

Solution to question 2

(i)(a). We say that the function $f : [a, b] \to \mathbb{R}$ achieves a *global maximum* at the point $x_0 \in [a, b]$ if $f(x) \le f(x_0)\,,\ \ \forall x \in [a, b]$. The number $f(x_0)$ is called the *maximum value* of f and is denoted $\max\limits_{x\in[a,b]} f(x)$. **[2 marks; bookwork]**

(i)(b). We call the point $x_0 \in (a, b)$ *critical* if $f'(x_0) = 0$. **[1 mark; bookwork]**

(i)(c). Suppose $f'(x_0) > 0$. Consider the function

$$g(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if} \quad x \neq x_0, \\ f'(x_0) & \text{if} \quad x = x_0. \end{cases}$$

This function is continuous at the point x_0, so by the Inertia Principle $\exists \delta > 0$ such that

$$|x - x_0| < \delta \quad \& \quad x \in [a, b] \qquad \Longrightarrow \qquad g(x) > 0.$$

Choose an arbitrary $\tilde{x} \in [a, b]$ such that $0 < \tilde{x} - x_0 < \delta$. Then

$$g(\tilde{x}) = \frac{f(\tilde{x}) - f(x_0)}{\tilde{x} - x_0} > 0$$

which implies $f(\tilde{x}) > f(x_0)$, contradicting the definition of a global maximum.
Similarly, the assumption $f'(x_0) < 0$ also leads to a contradiction.
This leaves us with $f'(x_0) = 0$ as the only possibility. **[9 marks; bookwork]**

(ii). The Attainment of Bounds Theorem and (i)(c) imply

Theorem 1 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then the global maximum (minimum) of f is achieved either at the endpoints or at interior critical points.*

We find the required maxima using the above Theorem. We denote $f(x) := \frac{x}{2} + \cos x$.

(ii)(a). The interior critical points are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ so

$$\begin{aligned}\max_{x\in[0,\pi]} f(x) &= \max\left\{f(0), f\left(\frac{\pi}{6}\right), f\left(\frac{5\pi}{6}\right), f(\pi)\right\} \\ &= \max\left\{1, \frac{\pi}{12} + \frac{\sqrt{3}}{2}, \frac{5\pi}{12} - \frac{\sqrt{3}}{2}, \frac{\pi}{2} - 1\right\} = \frac{\pi}{12} + \frac{\sqrt{3}}{2}.\end{aligned}$$

Note that the numerical values of the above four numbers are $\{1\,,\, 1.13\,,\, 0.44\,,\, 0.57\}$. Students are expected to be able to establish, without a calculator, that $\frac{\pi}{12} + \frac{\sqrt{3}}{2} > 1$. **[8 marks; unseen exercise]**

(ii)(b). The function is strictly increasing on $[-\pi, 0]$ so

$$\max_{x\in[-\pi,0]} f(x) = f(0) = 1.$$

[5 marks; unseen exercise]

Solution to question 3

(i).

Theorem 2 *If $f : (a, b) \to \mathbb{R}$ is a strictly monotone continuous function which is differentiable at $x_0 \in (a, b)$ with $f'(x_0) \neq 0$, then the function f^{-1} is differentiable at $f(x_0)$ and*

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}\,.$$

[5 marks; bookwork]

(ii)(a). The function $\tan : (-\pi/2,\, \pi/2) \to \mathbb{R}$ is strictly increasing and differentiable, and its derivative is non-zero. By the Theorem about the differentiability of the inverse function its inverse, $\arctan : \mathbb{R} \to (-\pi/2,\, \pi/2)$, is also differentiable and

$$\arctan'(\tan x) = \frac{1}{\tan' x} = \frac{1}{\frac{1}{\cos^2 x}} = \frac{1}{1+\tan^2 x}\,.$$

Setting $x = \arctan y$ we rewrite the above formula as $\arctan' y = \dfrac{1}{1+y^2}$. Replacing y by x we get $\arctan' x = \dfrac{1}{1+x^2}$. **[8 marks; question from an exercise sheet]**

(ii)(b). We are dealing with a geometric progression so

$$\arctan' x = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

The radius of convergence is 1. **[2 marks; unseen exercise]**

(ii)(c). Consider the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

In order to determine its radius of convergence, fix an $x \neq 0$ and apply the ratio test:

$$\frac{\left|\frac{(-1)^{k+1}}{2k+3} x^{2k+3}\right|}{\left|\frac{(-1)^k}{2k+1} x^{2k+1}\right|} = \frac{2k+1}{2k+3}|x^2| \to |x|^2$$

as $k \to \infty$. Hence, the power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is 1. By a theorem from the lecture course the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ is differentiable and can be differentiated term-by-term within the interval of convergence, so

$$g'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(x^{2k+1}\right)' = \sum_{k=0}^{\infty} (-1)^k x^{2k} = \arctan' x$$

on $(-1,1)$. **[6 marks; unseen exercise]**

(ii)(d). Consider the function $h(x) := \arctan x - g(x)$. Let us prove that $h(x) = 0$, $\forall x \in \mathbb{R}$. For $x = 0$ we have $h(0) = \arctan 0 - g(0) = 0 - 0 = 0$.
For $x \neq 0$ we apply the Mean Value Theorem:

$$h(x) = h'(\xi)\, x$$

for some ξ strictly between 0 and x. But by (ii)(c) $h'(\xi) = 0$, so $h(x) = 0$.

Thus, the power series for $\arctan x$ is

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

[4 marks; unseen exercise]

Solution to question 4

(i).

Theorem 3 *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that*

$$(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)\,.$$

[6 marks; bookwork]

(ii). Fix $x > 0$ and consider function

$$g(t) := f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - \dots - \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x-t)^n\,,$$

where t is the independent variable. Then $g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$. Applying Cauchy's Generalisation of the Mean Value Theorem to the functions $g(t)$ and $h(t) := (x-t)^{n+1}$ on the interval $[0,x]$ we get

$$\frac{0 - g(0)}{0 - x^{n+1}} = \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{-(n+1)(x-\xi)^n}$$

for some $\xi \in (0,x)$. Thus, $g(0) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$. It remains to note that $g(0) = f(x) - P_n(x) = R_n(x)$. **[11 marks; bookwork]**

(iii). Application of the result from (ii) to the function $f : (-1, 99) \to \mathbb{R}$, $f(x) = \ln(1+x)$ with $x = 1$ gives

$$\ln 2 = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} + R_n(1),$$

$$R_n(1) = \frac{(-1)^n}{(n+1)(1+\xi)^{n+1}}$$

for some $\xi \in (0,1)$. But

$$|R_n(1)| = \frac{1}{(n+1)(1+\xi)^{n+1}} \leq \frac{1}{(n+1)} \to 0 \qquad \text{as} \qquad n \to \infty$$

so

$$R_n(1) \to 0 \qquad \text{as} \qquad n \to \infty$$

which gives the required result. **[8 marks; the question itself is unseen but Taylor's formula for $\ln(1+x)$ was derived in the lectures as an example]**

Solution to question 5

(i)(a).

$$\begin{aligned} \underline{\int}_a^b f(x)\,dx &= \sup_P L(f,P), \\ \overline{\int}_a^b f(x)\,dx &= \inf_P U(f,P), \end{aligned}$$

where P is a partition of the interval $[a,b]$ and $L(f,P)$ and $U(f,P)$ lower and upper Darboux sums respectively. **[4 marks; bookwork]**

(i)(b).

Lemma 1 *Let P be a partition and let P' be another partition with one extra point. Then*

$$L(f,P) \le L(f,P'), \qquad U(f,P') \le U(f,P).$$

Proof Write the original partition P as

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the partition P' as

$$a = x_0 < x_1 < \cdots < x_{k-1} < x' < x_k < \cdots < x_{n-1} < x_n = b$$

Then

$$\begin{aligned} L(f,P') - L(f,P) = \left(\inf_{x\in[x_{k-1},x']} f(x)\right)(x'-x_{k-1}) + \left(\inf_{x\in[x',x_k]} f(x)\right)(x_k - x') \\ - \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x_k - x_{k-1}). \end{aligned}$$

But

$$\inf_{x\in[x_{k-1},x']} f(x) \ge \inf_{x\in[x_{k-1},x_k]} f(x), \qquad \inf_{x\in[x',x_k]} f(x) \ge \inf_{x\in[x_{k-1},x_k]} f(x),$$

so

$$\begin{aligned} L(f,P') - L(f,P) \ge \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x'-x_{k-1}) + \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x_k - x') \\ - \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x_k - x_{k-1}) = 0. \end{aligned}$$

The inequality $U(f,P') \le U(f,P)$ is proved in a similar way. □

Lemma 2 *If the partition P' is a refinement of the partition P then*

$$L(f,P) \le L(f,P'), \qquad U(f,P') \le U(f,P).$$

Proof Repeated application of Lemma 1. □

Lemma 3 *For any two partitions P and Q*

$$L(f,P) \le U(f,Q).$$

Proof Let $R := P \cup Q$ be the common refinement of P and Q. Then, by Lemma 2,

$$L(f,P) \le L(f,R), \qquad U(f,R) \le U(f,Q).$$

But, obviously, $L(f,R) \le U(f,R)$, so

$$L(f,P) \le L(f,R) \le U(f,R) \le U(f,Q).$$

□

The inequality

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

is an immediate consequence of the definitions of upper and lower Riemann integrals (see (i)(a)) and Lemma 3. **[12 marks; bookwork]**

(i)(c). The function f is said to be Riemann integrable on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx\,.$$

In this case the common value of the upper and lower Riemann integrals is denoted by

$$\int_a^b f(x)\,dx$$

and is called the *Riemann integral* of f over $[a,b]$. **[2 marks; bookwork]**

(ii). Let ε be an arbitrary positive number. Then there is a finite number $N = N(\varepsilon)$ of points $x \in [0,1]$ such that $f(x) \ge \varepsilon$. Each of these points can belong to at most two subintervals and, hence, $U(f,P) \le \varepsilon + 2N\omega(P)$ where $\omega(P)$ is the width of the partition P. As the width can be chosen to be arbitrarily small we have $\overline{\int_0^1} f(x)\,dx \le \varepsilon$. As ε can be chosen to be arbitrarily small we have $\overline{\int_0^1} f(x)\,dx \le 0$. But as the function f is nonnegative we have $0 \le \underline{\int_0^1} f(x)\,dx$. By (i)(b) we get $\underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx = 0$. By (i)(c) we conclude that the function f is Riemann integrable on $[0,1]$ and that $\int_0^1 f(x)\,dx = 0$. **[7 marks; unseen exercise]**

Solution to question 6

(i)(a).

$$T_n - T_{n+1} = \int_n^{n+1} f(x)\,dx - f(n+1) = \int_n^{n+1} f(x)\,dx - \int_n^{n+1} f(n+1)\,dx = \int_n^{n+1} (f(x) - f(n+1))\,dx \ge 0.$$

[4 marks; bookwork]

(i)(b).

$$T_n = \sum_{k=1}^{n-1}\int_k^{k+1} f(k)\,dx + f(n) - \sum_{k=1}^{n-1}\int_k^{k+1} f(x)\,dx = \sum_{k=1}^{n-1}\int_k^{k+1}(f(k)-f(x))\,dx + f(n) \geq f(n).$$

[10 marks; bookwork]

(i)(c). The result from (i)(b) and the fact that f is nonnegative imply that $T_n \geq 0$, $\forall n \in \mathbb{N}$. Thus, we are looking at a decreasing (see (i)(a)) sequence $\{T_n\}$ which is bounded from below. Such a sequence has to converge. **[3 marks; bookwork]**

(i)(d). Follows from the definition of T_n and (i)(c). **[1 mark; bookwork]**

(ii). Firstly, rewrite our series as

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(\ln(n+1))^2}\,.$$

Introduce the function $f : [1,\infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{(x+1)(\ln(x+1))^2}$. By (i)(d) the above series converges if and only if

$$\int_1^{\infty} \frac{dx}{(x+1)(\ln(x+1))^2}$$

converges. But the convergence of the latter integral is established by explicit calculations:

$$\int_1^b \frac{dx}{(x+1)(\ln(x+1))^2} = \left(-\frac{1}{\ln(x+1)}\right)\Bigg|_{x=1}^{x=b} = \frac{1}{\ln 2} - \frac{1}{\ln(b+1)} \to \frac{1}{\ln 2} \qquad \text{as} \qquad b \to +\infty.$$

[7 marks; unseen exercise]