All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (i) Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$.
   (a) Explain, in terms of ε and δ, the meaning of the mathematical statement "the function f is continuous at the point x_0".
   (b) Explain the meaning of the mathematical statement "the function f is differentiable at the point x_0".
   (c) Prove that if f is differentiable at x_0, then f is continuous at x_0.

   (ii) Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, for some $n \in \mathbb{N}$. By using the definition from (i)(b) prove that f is differentiable and that $f'(x) = nx^{n-1}$.

   (iii) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

   $$f(x) = \sin\frac{1}{x} \quad \text{if} \quad x \neq 0,$$
   $$f(0) = 0.$$

   Prove that f is not differentiable at 0.

2. (i) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b).
   (a) Explain what it means for f to achieve a *global maximum* at a point $x_0 \in [a, b]$.
   (b) Explain what it means for $x_0 \in (a, b)$ to be a *critical point* of the function f.
   (c) Suppose that f achieves a global maximum at the point $x_0 \in (a, b)$. Prove that x_0 is a critical point.

   (ii) Find, with justification,
   (a) $\displaystyle\max_{x\in[0,\pi]} \left(\frac{x}{2} + \cos x\right)$,
   (b) $\displaystyle\max_{x\in[-\pi,0]} \left(\frac{x}{2} + \cos x\right)$.

3. (i) State the Theorem about the differentiability of an inverse function.

(ii) Consider the function

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

and its inverse

$$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

(a) Prove that $\arctan$ is differentiable and that $\arctan' x = \dfrac{1}{1+x^2}$.

(b) Expand $\arctan' x$ into a power series. What is the radius of convergence of this power series?

(c) Construct a power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence 1 such that the function $g : (-1,1) \to \mathbb{R}$, $g(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies the condition

$$g'(x) = \frac{1}{1+x^2}$$

on $(-1,1)$.

(d) Derive, with justification, the power series expansion for $\arctan x$.

4. (i) State Cauchy's Generalisation of the Mean Value Theorem.

(ii) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a,b) \to \mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

Prove that for any $0 < x < b$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$$

for some $\xi \in (0,x)$.

(iii) Prove that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

5. (i) Let $f : [a,b] \to \mathbb{R}$ be bounded.

(a) Define the lower Riemann integral $\underline{\int_a^b} f(x)\,dx$ and the upper Riemann integral $\overline{\int_a^b} f(x)\,dx$.

(b) Prove that $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$.

(c) Explain what it means for f to be Riemann integrable on $[a,b]$.

(ii) Consider the function $f : [0,1] \to \mathbb{R}$ defined as follows:

$f(x) = 0$ if x is irrational,
$f(0) = 0$,
$f\left(\frac{p}{q}\right) = \frac{1}{q}$ if p and q are natural numbers with no common factors.

Prove that f is Riemann integrable on $[0,1]$ and determine $\int_0^1 f(x)\,dx$.

6. (i) Let the function $f : [1,\infty) \to \mathbb{R}$ be nonnegative, decreasing and continuous, and let $T_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx$, $n \in \mathbb{N}$.

(a) Prove that the sequence $\{T_n\}$ is decreasing.

(b) Prove that $T_n \ge f(n)$, $n \in \mathbb{N}$.

(c) Prove that the sequence $\{T_n\}$ converges.

(d) Prove that $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

(ii) Prove that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.