The operator curl acting on a closed 3-manifold: spectral analysis

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Joint work with Matteo Capoferri and Giovanni Bracchi.

Beyond the Hodge Theorem: curl and asymmetric pseudo-differential projections, jointly with Capoferri. Preprint arXiv:2309.02015.

A microlocal pathway to spectral asymmetry: curl and the eta invariant, jointly with Capoferri. Preprint arXiv:2502.18307.

Higher order Weyl coefficients for the operator curl, jointly with Bracchi and Capoferri. In preparation.

Setting up the problem at hand

Our playing field: connected oriented closed Riemannian 3-manifold (M, g).

Object of study: operator

$$\text{curl} := *\mathrm{d}$$

acting on real-valued 1-forms.

Goal: study the spectrum of curl, i.e. study the eigenvalue problem

$$\text{curl}\, u = \lambda u.$$

Why am I interested in this problem?

1 Maxwell's equations on $M \times \mathbb{R}$ reduce to the spectral problem for curl. Argument fails if manifold has a boundary.

2 The spectral problem operator curl is physically meaningful. Think of a photon living in curved space.

3 To date there is only one publication devoted to the spectral theory of the operator curl acting on a closed Riemannian manifold: C.Bär, 2019.

4 The spectrum of curl is asymmetric about zero. Like with Dirac operator. Think particle/antiparticle. Photon and antiphoton?

5 Spectral asymmetry is a classical subject. Atiyah, Patodi, Singer, Hitchin, Gilkey, Pontryagin, Hirzebruch, Chern, Simons, Seeley ... Hirzebruch L-polynomials, Hirzebruch $\hat{A}$-polynomials, Pontryagin forms, Pontryagin classes ...

Why am I interested in this problem? Continued ...

6 The sign of an eigenvalue of curl has a physical meaning. Notions of electromagnetic chirality and polarisation. Operator feels difference between right-handedness and left-handedness.

7 The operator curl is not elliptic because $\operatorname{curl}\operatorname{grad} = 0$. This poses a challenge to an analyst.

8 The operator curl is more fundamental than the Dirac operator becuase it does not involve the concept of connection.

9 I am obsessed with trying to understand elementary particles. The photon is the simplest elementary particle.

Operator theoretic definition of curl

First order of business: choose Hilbert space sensibly.

Bär chose Ω^1, the Hilbert space of all real-valued 1-forms.

We choose $\delta\Omega^2$, the Hilbert space of all real-valued coexact 1-forms . 'Coexact' means 'divergence of a 2-form'.

Second order of business: choose domain sensibly.

We choose $\delta\Omega^2 \cap H^1$, where H^1 is the Sobolev space of real-valued 1-forms which are square integrable together with their first partial derivatives. So

$$\text{curl} = *\mathrm{d} : \delta\Omega^2 \cap H^1 \to \delta\Omega^2.$$

Theorem 1

(a) The operator curl is self-adjoint.
(b) The spectrum of curl is discrete and accumulates to $+\infty$ and to $-\infty$.
(c) Zero is not an eigenvalue of curl.
(d) The operator curl^{-1} is a bounded operator from $\delta\Omega^2 \cap H^s$ to $\delta\Omega^2 \cap H^{s+1}$ for all $s \geq 0$.

Notation for the eigensystem of curl:

$$(\lambda_j, u_j), \quad j = \pm 1, \pm 2, \ldots.$$

How does one measure spectral asymmetry?

Definition of *eta function:*

$$\eta_{\text{curl}}(s) := \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{\operatorname{sgn} \lambda_j}{|\lambda_j|^s} \,.$$

Series converges absolutely for $\operatorname{Re} s > 3$.

Meromorphic continuation to $\mathbb{C}$.

Eta function generalises the more familiar zeta function.

Definition of *eta invariant:* $\eta_{\text{curl}}(0)$.

Number of positive eigenvalues minus number of negative ones.

Number of particles minus number of antiparticles.

Our main result

We define eta invariant without resorting to analytic continuation.

We do it using pseudodifferential techniques.

What is a pseudodifferential operator?

A pseudodifferential operator can be written locally as

$$f(x) \mapsto (2\pi)^{-3} \int e^{(x-y)^\gamma \xi_\gamma} a(x,\xi) f(y) \, \mathrm{d}y \, \mathrm{d}\xi \, .$$

Here $a(x,\xi)$ is the *symbol* which admits an asymptotic expansion into components positively homogeneous in ξ. The leading term in this asymptotic expansion is called *principal symbol*.

Example of pseudodifferential operator: $\sqrt{-\Delta}$, where Δ is the Laplace–Beltrami operator. The principal symbol of the operator $\sqrt{-\Delta}$ is $\|\xi\|$, where $\|\xi\| := \sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\,\xi_\beta}\,$.

Technical issues.

- We are working on a 3-manifold as opposed to flat space $\mathbb{R}^3$.
- We are working with operators acting on 1-forms as opposed to scalar functions. Our symbols are matrix-functions.

Our approach to spectral asymmetry

Put

$$P_{\pm} := \theta(\pm\,\mathrm{curl})\,,$$

where

$$\theta(x) := \begin{cases} 1, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

is the Heaviside step function.

Then, morally,

$$\eta_{\mathrm{curl}}(0) = \mathrm{Tr}(P_{+} - P_{-})\,.$$

Major problem: operator $P_{+} - P_{-}$ is not of trace class.

Calculating the trace of an operator acting on 1-forms

Consider

$$Q: \ u_\alpha(x) \mapsto \int_M \mathfrak{q}_\alpha{}^\beta(x,y)\, u_\beta(y)\, \rho(y)\, \mathrm{d}y\,,$$

where $\mathfrak{q}$ is the (distributional) integral kernel (Schwartz kernel) and ρ is the Riemannian density.

Suppose that the integral kernel $\mathfrak{q}$ is sufficiently smooth. Then

$$\operatorname{Tr} Q = \int_M \mathfrak{q}_\alpha{}^\alpha(x,x)\, \rho(x)\, \mathrm{d}x\,,$$

Idea: split the process of calculating trace into two separate steps.

- Take matrix trace first, which would give a scalar operator.
- Calculate the trace of the scalar operator the usual way, by taking the value of the integral kernel on the diagonal $x = y$ and integrating over the manifold M.

Matrix trace of an operator acting on 1-forms

Definition 2 The matrix trace of an operator acting on 1-forms is the scalar operator obtained by contracting tensor indices in the integral kernel $\mathfrak{q}_\alpha{}^\beta(x,y)$ of the original operator. No assumptions on the smoothness of the integral kernel.

Slight problem: tensor indices α and β live at different points, x and y. To make above definition invariant need to perform parallel transport along shortest geodesic connecting x and y.

Another minor problem: need smooth cut-off about the diagonal $x = y$ so that the shortest geodesic connecting x and y is unique.

Matrix trace of an operator acting on 1-forms is defined uniquely modulo the addition of a scalar operator whose integral kernel is infinitely smooth and vanishes in a neighbourhood of the diagonal.

The asymmetry operator

Definition 3 The *asymmetry operator* A is defined as the matrix trace of the operator $P_+ - P_-$.

The asymmetry operator is a self-adjoint scalar pseudodifferential operator determined by the Riemannian 3-manifold (M, g) and its orientation.

Constructing the projection operators $P_\pm$

Theorem 4 The operators P_+ and P_- are pseudodifferential operators of order zero and we have written down **explicitly** the homogeneous components of their symbols of degree of homogeneity $0, -1, -2, -3$.

Algorithm leading to the determination of full symbols of pseudodifferential projections is described in our paper

Capoferri and Vassiliev, *Invariant subspaces of elliptic systems I: pseudodifferential projections*, Journal of Functional Analysis, 2022.

Algorithm is global and does not use local coordinates. Magic!

Implementation of 'magic' algorithm benefits from the use of the computer algebra package Mathematica©.

Where did the 'magic' algorithm come from?

Spectral theory of elliptic systems. Second Weyl coefficient.

1 V.Ivrii, 1980, Soviet Math. Doklady.
2 V.Ivrii, 1982, Funct. Anal. Appl.
3 G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.
4 V.Ivrii, 1984, Springer Lecture Notes. In 1998 book in Springer. In 2019 another book in 5 volumes, also in Springer.
5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.
6 W.J.Nicoll, PhD thesis, 1998, University of Sussex.
7 I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.
8 O.Chervova, R.J.Downes and D.Vassiliev, 2013, Journal of Spectral Theory.

2020: Matteo Capoferri and I realised that we have been looking at elliptic systems the wrong way. Should look for almost invariant subspaces and pseudodifferential projections. Benefit of hindsight.

The miracle

Theorem 5 The asymmetry operator is a pseudodifferential operator of order -3.

Reason for miracle: symmetries of the Riemannian curvature tensor

Corollary 6 The asymmetry operator is *almost* trace class.

Singularity of the integral kernel of the asymmetry operator

Theorem 7 The principal symbol of the asymmetry operator reads

$$A_{\mathrm{prin}}(x,\xi) = -\frac{\varepsilon^{\alpha\beta\gamma}}{2\,\rho(x)\,\|\xi\|^5}\,\nabla_\alpha \operatorname{Ric}_\beta{}^\delta(x)\,\xi_\gamma\,\xi_\delta\,,$$

where Ric is the Ricci curvature tensor, $\nabla \operatorname{Ric}$ is its covariant derivative and ε is the totally antisymmetric symbol (Levi-Civita symbol), $\varepsilon^{123} := +1$.

Corollary 8 The singularity of the integral kernel $\mathfrak{a}(x,y)$ of the asymmetry operator is very weak. Namely, $\mathfrak{a}(x,y)$ is a bounded function, smooth outside the diagonal and discontinuous on the diagonal: for any $x \in M$ the limit $\lim_{y\to x}\mathfrak{a}(x,y)$ depends on the direction along which y tends to x.

The regularised local trace of the asymmetry operator

Denote by $\mathbb{S}_\epsilon(x)$ the geodesic sphere of radius $\epsilon > 0$ centred at the point $x \in M$.

Theorem 9 For any $x \in M$ the limit

$$\lim_{\epsilon \to 0^+} \frac{1}{4\pi\epsilon^2} \int_{\mathbb{S}_\epsilon(x)} \mathfrak{a}(x,y)\, \mathrm{d}S_y$$

exists and defines a scalar continuous function

$$\psi_{\mathrm{curl}}^{\mathrm{loc}}(x)\,, \quad \psi_{\mathrm{curl}}^{\mathrm{loc}} : M \to \mathbb{R}\,.$$

Definition 10 We call $\psi_{\mathrm{curl}}^{\mathrm{loc}}(x)$ *the regularised local trace of the asymmetry operator*.

The regularised global trace of the asymmetry operator

Definition 11 We call the number

$$\psi_{\mathsf{curl}} := \int_M \psi_{\mathsf{curl}}^{\mathrm{loc}}(x)\,\rho(x)\,\mathrm{d}x$$

the regularised global trace of the asymmetry operator.

Reconciling our approach with the classical one

Using microlocal techniques, we have defined a differential geometric invariant ψ_{curl}, a measure of the asymmetry of our Riemannian manifold under change of orientation.

Is it true that $\psi_{\mathrm{curl}} = \eta_{\mathrm{curl}}(0)$?

Theorem 12 Yes, it is true that $\psi_{\mathrm{curl}} = \eta_{\mathrm{curl}}(0)$.

Proof in our second preprint.

Spectral asymptotics

Subject of our third paper, the one in preparation.

Two counting functions

$$N^{\pm}(\lambda) := \begin{cases} 0 & \text{for } \lambda \le 0, \\ \sum_{0<\pm\lambda_j<\lambda} 1 & \text{for } \lambda > 0. \end{cases}$$

Seeking asymptotic expansions

$$N^{\pm}(\lambda) \stackrel{?}{=} a_3^{\pm}\lambda^3 + a_2^{\pm}\lambda^2 + a_1^{\pm}\lambda + \dots \quad \text{as} \quad \lambda \to +\infty.$$

Here the coefficients are called *Weyl coefficients*.

Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, Amer. Math. Soc., Providence (RI), 1997.

I have a dream

My dream is to prove that

$$\eta_{\text{curl}}(-1) = 0.$$

Morally, this would mean that

$$\sum_{j \in \mathbb{Z} \setminus \{0\}} \lambda_j = 0.$$

'The operator curl is trace-free'.

Compare with famous formula

$$1 + 2 + 3 + \cdots = -\frac{1}{12}\,.$$