Spectral theory of differential operators: what's it all about and what is its use
Part V

Dmitri Vassiliev

University College London

June 2024

Spectral theory for type 2 systems

Looking at a self-adjoint elliptic system of m PDEs, each PDE first order, on a compact d-dimensional manifold M without boundary. Semi-boundedness is not assumed. Think particle/antiparticle.

Principal symbol is an $m \times m$ positive Hermitian matrix-function on $T^*M \setminus \{0\}$, non-degenerate but not necessarily sign-definite.

I assume that the eigenvalues of the principal symbol are simple.

Example of a type 2 system: massless Dirac operator

System of two first order PDEs for two complex-valued unknowns on an oriented Riemannian 3-manifold.

Counting eigenvalues

Need to count positive and negative eigenvalues separately.

Two counting functions:

$$N_\pm(\lambda) := \begin{cases} 0 & \text{for } \lambda \leq 0, \\ \sum_{0<\pm\lambda_k<\lambda} 1 & \text{for } \lambda > 0. \end{cases}$$

Eigenvalues of the principal symbol are the new Hamiltonians.

Formula for the first Weyl coefficient requires minor modification.

Important: the second Weyl coefficient comes from M itself.

But evaluation of the second Weyl coefficient is not easy.

Timeline: evaluating the second Weyl coefficient

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.

4 V.Ivrii, 1984, Springer Lecture Notes. In 1998 book in Springer. In 2019 another book in 5 volumes, also in Springer.

5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

6 W.J.Nicoll, PhD thesis, 1998, University of Sussex.

7 I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.

8 O.Chervova, R.J.Downes and D.Vassiliev, 2013, Journal of Spectral Theory.

Final chapter in the saga: invariant subspaces

Capoferri and Vassiliev, *Invariant subspaces of elliptic systems I: pseudodifferential projections*, Journal of Functional Analysis, 2022.

Capoferri and Vassiliev, *Invariant subspaces of elliptic systems II: spectral theory*, Journal of Spectral Theory, 2022.

Any (OK, almost any) system of m (pseudo)differential equations can be associated with m almost invariant almost orthogonal subspaces of the original Hilbert space.

Projections onto these subspaces are pseudodifferential operators, and we have an explicit algorithm for the determination of their full symbols. Algorithm is global and does not use local coordinates.

Have been looking at systems the wrong way. Should have been seeking invariant subspaces and projections. Benefit of hindsight.

Branching out into some **very** serious geometry

Eta function of an operator A:

$$\eta_A(s) := \sum_k \frac{\operatorname{sgn} \lambda_k}{|\lambda_k|^s} \,.$$

Note: series converges absolutely for $\operatorname{Re} s > d$.

Eta invariant of an operator A: $\eta_A(0)$.

Number of positive eigenvalues minus number of negative ones.

We can tackle the eta invariant without analytic continuation:

$$\eta_A(0) = \operatorname{Tr}[\theta(A) - \theta(-A)].$$

Contributors to this subject area: Michael Atiyah, Vijay Kumar Patodi, Isadore Singer, Nigel Hitchin, Peter Gilkey, Lev Pontryagin, Friedrich Hirzebruch, Shiing-Shen Chern, Jim Simons, Bob Seeley.