Spectral theory of differential operators: what's it all about and what is its use
Part IV

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Spectral theory for type 1 systems

Looking at a self-adjoint elliptic system of m PDEs, each of even order $2n$, on a compact d-dimensional manifold M with boundary ∂M. Requires mn boundary conditions. I assume that the system is semi-bounded from below

Principal symbol is an $m \times m$ positive Hermitian matrix-function on $T^*M \setminus \{0\}$.

I assume that the eigenvalues of the principal symbol have constant multiplicity.

Extension of results from scalar case to type 1 systems is pretty straightforward.

Take eigenvalues of the principal symbol and extract $(2n)^{\mathrm{th}}$ positive root. These are the new Hamiltonians.

Reflection law: allow jumps from one Hamiltonian to another.

Formula for the first Weyl coefficient requires minor modification.

Algorithm for the evaluation of the second Weyl coefficient remains the same.

Important: the second Weyl coefficient comes from the boundary ∂M, as in the scalar case. Contributions to the second Weyl coefficient from M itself cancel out due to some symmetries.

Example of a type 1 system: linear elasticity

Equations of linear elasticity were first formulated by Baron Augustin-Louis Cauchy in 1828–29.

Variational formulation of linear elasticity

Quadratic functional

$$\mathcal{E}[\mathbf{u}] := \int_\Omega \left(\lambda\,(\nabla_\alpha u^\alpha)^2 + \mu\,(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha)\,\nabla^\alpha u^\beta\right)\,\sqrt{\det g}\;dx\,,$$

where λ and μ are real constants called *Lamé coefficients* which are assumed to satisfy

$$\mu > 0, \qquad d\lambda + 2\mu > 0,$$

$\mathbf{u}$ is the vector field of displacements (unknown quantity), ∇ is the Levi-Civita connection and $\sqrt{\det g}$ is the Riemannian density.

Will have to denote spectral parameter by Λ.

Variation of quadratic functional gives spectral problem

$$\mathcal{L}\mathbf{u} = \Lambda\mathbf{u},$$

where

$$(\mathcal{L}\mathbf{u})^\alpha := -\mu\left(\nabla_\beta\nabla^\beta u^\alpha + \mathrm{Ric}^\alpha{}_\beta u^\beta\right) - (\lambda+\mu)\nabla^\alpha\nabla_\beta u^\beta.$$

Principal symbol has two eigenvalues: simple eigenvalue

$$(\lambda + 2\mu)\|\xi\|^2$$

and eigenvalue

$$\mu\|\xi\|^2$$

of multiplicity $d-1$. These correspond to longitudinal and transverse elastic waves, respectively. Waves mix up when reflected from the boundary, giving us a branching Hamiltonian billiards.

Boundary conditions

Dirichlet condition

$$\mathbf{u}|_{\partial\Omega} = 0$$

or free boundary condition

$$\mathcal{T}\mathbf{u}|_{\partial\Omega} = 0$$

where $\mathcal{T}$ is the boundary traction operator defined by

$$(\mathcal{T}\mathbf{u})^\alpha := \lambda n^\alpha \nabla_\beta u^\beta + \mu \left(n^\beta \nabla_\beta u^\alpha + n_\beta \nabla^\alpha u^\beta \right).$$

Important: the free boundary condition is not the Neumann boundary condition $n^\beta \nabla_\beta u^\alpha = 0$.

In 1885 Lord Rayleigh analysed the free boundary condition and discovered *Rayleigh waves*.

Timeline of spectral analysis of linear elasticity

1912: P.Debye writes down first Weyl coefficient.

1915: H.Weyl provides rigorous proof of Debye's result.

1950: E.W.Montroll, incorrect calculation of the second Weyl coefficient.

1960: M.Dupuis, R.Mazo, and L.Onsager write down second Weyl coefficient for $d = 3$, both for Dirichlet and free boundary.

1997: I check the results of M.Dupuis, R.Mazo, and L.Onsager for $d = 3$ using my algorithm, and also deal with $d = 2$.

2023: M.Capoferri, L.Friedlander, M.Levitin, and D.Vassiliev. Second Weyl coefficient for any dimension. For odd dimensions explicit formulae in terms of Lamé parameters, for even dimensions formulae in terms of elliptic integrals.

Two-Term Spectral Asymptotics in Linear Elasticity

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Matteo Capoferri, Leonid Friedlander, Michael Levitin & Dmitri Vassiliev

Abstract

We establish the two-term spectral asymptotics for boundary value problems of linear elasticity on a smooth compact Riemannian manifold of arbitrary dimension. We also present some illustrative examples and give a historical overview of the subject. In particular, we correct erroneous results published by Liu (J Geom Anal 31:10164–10193, 2021).

Elasticity operator in $\mathbb{R}^2$:

$$\mathcal{L} : \begin{pmatrix} u^1(x,y) \\ u^2(x,y) \end{pmatrix} \mapsto - \begin{pmatrix} (\lambda + 2\mu)\partial_{xx}^2 + \mu\partial_{yy}^2 & (\lambda + \mu)\partial_{xy}^2 \\ (\lambda + \mu)\partial_{xy}^2 & \mu\partial_{xx}^2 + (\lambda + 2\mu)\partial_{yy}^2 \end{pmatrix} \begin{pmatrix} u^1(x,y) \\ u^2(x,y) \end{pmatrix}$$

Reflection about x-axis:

$$\mathcal{R} : \begin{pmatrix} u^1(x,y) \\ u^2(x,y) \end{pmatrix} \mapsto \begin{pmatrix} u^1(x,-y) \\ u^2(x,-y) \end{pmatrix}.$$

Chain Rule tells us that

$$[\mathcal{L}, \mathcal{R}] \neq 0.$$

arXiv > math > arXiv:2208.02679

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Two-term spectral asymptotics in linear elasticity on a Riemannian manifold

Genqian Liu

In this note, by explaining two key methods that were employed in \cite{Liu-21} and by giving some remarks, we show that the proof of Theorem 1.1 in \cite{Liu-21} is a rigorous proof based on theory of strongly continuous semigroups and pseudodifferential operators. All remarks and comments to paper \cite{Liu-21}, which were given by Matteo Capoferri, Leonid Friedlander, Michael Levitin and Dmitri Vassiliev in \cite{CaFrLeVa-22}, are incorrect. The so-called "numerical counter-examples" in \cite{CaFrLeVa-22} are useless examples for the two-term asymptotics of the counting functions of the elastic eigenvalues. Clearly, the conclusion and the proof of \cite{Liu-21} are completely correct.

[Submitted on 23 May 2023 (v1), last revised 24 Dec 2023 (this version, v6)]

Remarks on paper "Two-term spectral asymptotics in linear elasticity"

Genqian Liu

In this note, we shall point out that all ``numerically calculations'' and figures in \cite{CaFrLeVa-23} are wrong because these calculations are based on some incorrect formulas. Furthermore, by pointing out several serious errors in \cite{CaFrLeVa-23} and especially by Section 7, Proposition 7.1, Remarks 7.2--7.3, and Section 8 (a result of A. Pierzchalski and B. Ørsted) we show that the conclusions published by Matteo Capoferri, Leonid Friedlander, Michael Levitin and Dmitri Vassiliev (J Geom Anal (2023)33:242) as well as the main ``algorithm'' theory of the book \cite{SaVa-97} are completely wrong. Finally, we explain the correctness of proof of Theorem 1.1 in our paper \cite{Liu-21} by giving some remarks and putting the whole proof in Appendix (see also \cite{Liu-22b} and \cite{Liu-22c}).

[Submitted on 14 Sep 2023 (v1), last revised 9 Feb 2024 (this version, v3)]

On an algorithm for two-term spectral asymptotic formulas

Genqian Liu

In the book [Yu. Safarov and D. Vassiliev, The asymptotic distribution of eigenvalues of partial differential operators, Amer. Math. Soc., Providence, RI, 1997], a key and central ``algorithm'' was established, by which the coefficients of two-term asymptotic expansions of the eigenvalue counting functions can be explicitly calculated for many partial differential operators under an additional geometric assumption. In this paper, we give a counter-example to this ``algorithm'' by discussing the case of elastic eigenvalues. This implies that the most conclusions in the above book written by Yu. Safarov and D. Vassiliev are fundamentally wrong because they are based on the erroneous ``algorithm''.