Beyond the Hodge Theorem: curl and asymmetric pseudodifferential projections

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What is spectral asymmetry?

Let Q be an Hermitian matrix.

Positive eigenvalues λ_k interpreted as particles.

Negative eigenvalues λ_k interpreted as antiparticles.

Definition of *eta invariant:*

$$\#\{k \,|\, \lambda_k > 0\} - \#\{k \,|\, \lambda_k < 0\}.$$

Number of positive eigenvalues minus the number of negative eigenvalues.

Number of particles minus the number of antiparticles.

Let Q be a self-adjoint operator in a Hilbert space. Suppose the spectrum of Q is discrete.

Definition of *eta function:*

$$\eta_Q(s) := \sum_{\lambda_k \neq 0} \frac{\operatorname{sgn} \lambda_k}{|\lambda_k|^s} \,.$$

Series converges absolutely for sufficiently large positive $\operatorname{Re} s$.

Meromorphic continuation to $\mathbb{C}$.

Eta function generalises the more familiar zeta function.

Definition of *eta invariant:* $\eta_Q(0)$.

Mathematicians who contributed to this subject area: Atiyah, Patodi, Singer, Hitchin, Gilkey, Pontryagin, Hirzebruch, Chern, Simons, Seeley ...

Hirzebruch L-polynomials, Hirzebruch $\hat{A}$-polynomials, Pontryagin forms, Pontryagin classes ...

Our main result

We define eta invariant without resorting to analytic continuation.

We do it using pseudodifferential techniques.

The operator curl

Work on connected oriented closed Riemannian 3-manifold (M, g).

$\Omega^1(M)$ is the space of real-valued 1-forms on M.

The operator

$$\mathrm{curl} : \Omega^1(M) \to \Omega^1(M)$$

acts on 1-forms and is defined by the formula

$$\mathrm{curl} := *\mathrm{d}\,.$$

Why is the operator curl interesting?

1 Homogeneous vacuum Maxwell equations on $M \times \mathbb{R}$ reduce to

$$\left(-i\frac{\partial}{\partial t} + \operatorname{curl}\right) u = 0,$$

$$\operatorname{div} u = 0,$$

where $u := E + iB$. Think of a single photon living in curved space.

2 The spectrum of curl is asymmetric. Eigenvalues accumulate to $+\infty$ and to $-\infty$, but in an asymmetric fashion. Operator feels difference between right-handedness and left-handedness.

3 The sign of eigenvalues of curl has a physical meaning. Electromagnetic chirality. Photon and antiphoton?

4 Very few publications on the spectrum of curl. Paper by Jason Lotay (2012) and another by Christian Bär (2019).

Why is the study of the operator curl challenging?

Operator is not elliptic.

Determinant of principal symbol of curl is identically zero.

Eigenvalues of the (principal) symbol of the operator curl read

$$0 \qquad \text{and} \qquad \pm\|\xi\|,$$

where ξ is momentum (dual variable to the position variable x) and $\|\xi\| := \sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta}\,.$

Proper definition of curl and its basic properties

It is natural to define curl as an operator

$$\mathrm{curl} : \delta\Omega^2 \cap H^1 \to \delta\Omega^2,$$

where $\delta\Omega^2$ is the Hilbert space of real-valued coexact 1-forms with inner product

$$\langle u, v\rangle := \int_M *u \wedge v = \int_M u \wedge *v$$

and H^1 is the Sobolev space of 1-forms which are square integrable together with their first partial derivatives.

Theorem 1

(a) The operator curl is self-adjoint.

(b) The spectrum of curl is discrete and accumulates to $+\infty$ and to $-\infty$.

(c) Zero is not an eigenvalue of curl.

(d) The operator curl^{-1} is a bounded operator from $\delta\Omega^2 \cap H^s$ to $\delta\Omega^2 \cap H^{s+1}$ for all $s \geq 0$.

Our approach to spectral asymmetry

Put

$$P_\pm := \theta(\pm\,\mathrm{curl})\,,$$

where

$$\theta(x) := \begin{cases} 1, & x > 0, \\ 0, & x \le 0, \end{cases}$$

is the Heaviside step function.

Then, morally,

$$\eta_{\mathrm{curl}}(0) = \mathrm{Tr}(P_+ - P_-)\,.$$

Major problem: operator $P_+ - P_-$ is not of trace class.

Notion of pseudodifferential operator

Pseudodifferential operator P of order s acting on 1-forms over a manifold of dimension d. Representation in local coordinates

$$u_\mu(x) \mapsto \frac{1}{(2\pi)^d} \int p_\mu{}^\nu(x,\xi)\, e^{i(x-y)^\alpha \xi_\alpha} u_\nu(y)\, dy\, d\xi\,.$$

In our case $d = 3$ and $s = 0$.

Here $p_\mu{}^\nu(x,\xi)$ is the *symbol* which admits an asymptotic expansion into components positively homogeneous in momentum ξ:

$$p_\mu{}^\nu(x,\xi) \sim [p_s]_\mu{}^\nu(x,\xi) + [p_{s-1}]_\mu{}^\nu(x,\xi) + \ldots.$$

The subscripts $s, s-1, \ldots,$ indicate the degree of homogeneity.

Invariant calculus for pseudodifferential operators acting on 1-forms

Definition of subprincipal symbol for operators acting on 1-forms.

Original definition of subprincipal symbol for scalar operators acting on half-densities is due to Duistermaat and Hörmander (1972).

Subprincipal symbol of adjoint.

Subprincipal symbol of composition.

In dimension d the subprincipal symbol of Hodge Laplacian is zero.

In dimension 3 the subprincipal symbol of curl is zero.

Calculating the trace of an operator acting on 1-forms

Consider

$$Q: \ u_\alpha(x) \mapsto \int_M \mathfrak{q}_\alpha{}^\beta(x,y)\, u_\beta(y)\, \rho(y)\, \mathrm{d}y\,,$$

where $\mathfrak{q}$ is the (distributional) integral kernel (Schwartz kernel) and ρ is the Riemannian density.

Suppose that the integral kernel $\mathfrak{q}$ is sufficiently smooth. Then

$$\operatorname{Tr} Q = \int_M \mathfrak{q}_\alpha{}^\alpha(x,x)\, \rho(x)\, \mathrm{d}x\,,$$

Idea: split the process of calculating trace into two separate steps.

- Take matrix trace first, which would give a scalar operator.
- Calculate the trace of the scalar operator the usual way, by taking the value of the integral kernel on the diagonal $x = y$ and integrating over the manifold M.

Matrix trace of an operator acting on 1-forms

Definition 2 The matrix trace of an operator acting on 1-forms is the scalar operator obtained by contracting tensor indices in the integral kernel ${\mathfrak q_\alpha}^\beta(x,y)$ of the original operator. No assumptions on the smoothness of the integral kernel.

Slight problem: tensor indices α and β live at different points, x and y. To make above definition invariant need to perform parallel transport along shortest geodesic connecting x and y.

Another minor problem: need smooth cut-off about the diagonal $x=y$ so that the shortest geodesic connecting x and y is unique.

Matrix trace of an operator acting on 1-forms is defined uniquely modulo the addition of a scalar operator whose integral kernel is infinitely smooth and vanishes in a neighbourhood of the diagonal.

Properties of matrix trace of operator acting on 1-forms

Matrix trace of adjoint.

Principal and subprincipal symbols of matrix trace.

Matrix trace of a differential operator is a differential operator.

In dimension d the matrix trace of the Hodge Laplacian is $d\Delta + \frac{1}{3}\,\mathrm{Sc}$, where Δ is the Laplace–Beltrami operator and Sc is scalar curvature.

In dimension 3 the matrix trace of curl is zero.

In dimension 3 the matrix trace of curl^3 is zero.

The asymmetry operator

Definition 3 The *asymmetry operator* A is defined as the matrix trace of the operator $P_+ - P_-$.

The asymmetry operator is a self-adjoint scalar pseudodifferential operator determined by the Riemannian 3-manifold (M, g) and its orientation.

Constructing the projection operators $P_\pm$

Theorem 4 The operators P_+ and P_- are pseudodifferential operators of order zero and we have written down **explicitly** the homogeneous components of their symbols of degree of homogeneity $0, -1, -2, -3$.

Algorithm leading to the determination of full symbols of pseudodifferential projections is described in our paper

Capoferri and Vassiliev, *Invariant subspaces of elliptic systems I: pseudodifferential projections*, Journal of Functional Analysis, 2022.

Algorithm is global and does not use local coordinates. Magic!

Implementation of 'magic' algorithm benefits from the use of the computer algebra package Mathematica©.

Where did the 'magic' algorithm come from?

Spectral theory of elliptic systems. Second Weyl coefficient.

1 V.Ivrii, 1980, Soviet Math. Doklady.
2 V.Ivrii, 1982, Funct. Anal. Appl.
3 G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.
4 V.Ivrii, 1984, Springer Lecture Notes. In 1998 book in Springer. In 2019 another book in 5 volumes, also in Springer.
5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.
6 W.J.Nicoll, PhD thesis, 1998, University of Sussex.
7 I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.
8 O.Chervova, R.J.Downes and D.Vassiliev, 2013, Journal of Spectral Theory.

2020: Matteo Capoferri and I realised that we have been looking at elliptic systems the wrong way. Should look for almost invariant subspaces and pseudodifferential projections. Benefit of hindsight.

The miracle

Theorem 5 The asymmetry operator is a pseudodifferential operator of order -3.

Reason for miracle: symmetries of the Riemannian curvature tensor

Corollary 6 The asymmetry operator is *almost* trace class.

Singularity of the integral kernel of the asymmetry operator

Theorem 7 The principal symbol of the asymmetry operator reads

$$A_{\mathrm{prin}}(x,\xi) = -\frac{\varepsilon^{\alpha\beta\gamma}}{2\,\rho(x)\,\|\xi\|^5}\,\nabla_\alpha \operatorname{Ric}_\beta{}^\delta(x)\,\xi_\gamma\,\xi_\delta\,,$$

where Ric is the Ricci curvature tensor, $\nabla\operatorname{Ric}$ is its covariant derivative and ε is the totally antisymmetric symbol (Levi-Civita symbol), $\varepsilon^{123}:=+1$.

Corollary 8 The singularity of the integral kernel $\mathfrak{a}(x,y)$ of the asymmetry operator is very weak. Namely, $\mathfrak{a}(x,y)$ is a bounded function, smooth outside the diagonal and discontinuous on the diagonal: for any $x\in M$ the limit $\lim_{y\to x}\mathfrak{a}(x,y)$ depends on the direction along which y tends to x.

The regularised local trace of the asymmetry operator

Denote by $\mathbb{S}_\epsilon(x)$ the geodesic sphere of radius $\epsilon > 0$ centred at the point $x \in M$.

Theorem 9 For any $x \in M$ the limit

$$\lim_{\epsilon \to 0^+} \frac{1}{4\pi\epsilon^2} \int_{\mathbb{S}_\epsilon(x)} \mathfrak{a}(x,y)\, \mathrm{d}S_y$$

exists and defines a scalar continuous function

$$\psi^{\mathrm{loc}}_{\mathrm{curl}}(x)\,, \quad \psi^{\mathrm{loc}}_{\mathrm{curl}} : M \to \mathbb{R}\,.$$

Definition 10 We call $\psi^{\mathrm{loc}}_{\mathrm{curl}}(x)$ *the regularised local trace of the asymmetry operator*.

The regularised global trace of the asymmetry operator

Definition 11 We call the number

$$\psi_{\mathsf{curl}} := \int_M \psi^{\mathrm{loc}}_{\mathsf{curl}}(x)\,\rho(x)\,\mathrm{d}x$$

the regularised global trace of the asymmetry operator.

Reconciling our approach with the classical one

Using microlocal techniques, we have defined a differential geometric invariant ψ_{curl}, a measure of the asymmetry of our Riemannian manifold under change of orientation.

Is it true that $\psi_{\mathrm{curl}} = \eta_{\mathrm{curl}}(0)$?

Yes, it is.

Capoferri and Vassiliev, *A microlocal pathway to spectral asymmetry: curl and the eta invariant*, in preparation.