Two-term spectral asymptotics in linear elasticity

Dmitri Vassiliev

9 May 2023

Cornell University

We gratefully acknowledge support from the Simons Foundation and member institutions.

arXiv > math > arXiv:2207.13636

Search... All fields Search

Help | Advanced Search

Mathematics > Spectral Theory

[Submitted on 27 Jul 2022]

Two-term spectral asymptotics in linear elasticity

Matteo Capoferri, Leonid Friedlander, Michael Levitin, Dmitri Vassiliev

Motivated in part by the erroneous results in "Geometric invariants of spectrum of the Navier-Lamé operator" by Genqian Liu published in the Journal of Geometric Analysis 31 (2021), 10164--10193, we establish the two-term spectral asymptotics for boundary value problems of linear elasticity on a smooth compact Riemannian manifold of arbitrary dimension. We also present some illustrative examples and give a historical overview of the subject.

Comments:	37 pages, 10 figures
Subjects:	**Spectral Theory (math.SP)**; Analysis of PDEs (math.AP); Differential Geometry (math.DG)
MSC classes:	35P20, 35Q74, 74J05
Cite as:	arXiv:2207.13636 **[math.SP]**

Download:

- PDF
- Other formats

(license)

Current browse context:
math.SP
< prev | next >
new | recent | 2207

Change to browse by:
math
math.AP
math.DG

References & Citations

- NASA ADS
- Google Scholar
- Semantic Scholar

Export BibTeX Citation

May

May 31	**Maciej Zworski** Spectral theory of internal waves in fluids Video
May 24	**Genqian Liu** Geometric invariants of spectrum of the Navier-Lame operator Video
May 17	**Alexandre Girouard** Free boundary minimal surfaces and large Steklov eigenvalues Video
May 10	**Ram Band** Neumann domains on manifolds and graphs Video
May 3	**Jared Wunsch** Semiclassical analysis and the convergence of the finite element method Video

Winter 2021

BBC - Home | Facebook | La forza del destino - Wikipedia

en.wikipedia.org/wiki/La_forza_del_destino

Wikipedia
The Free Encyclopedia

Search Wikipedia

Create account Log in

La forza del destino

39 languages

Article Talk Read Edit View history Tools

Contents [hide]

From Wikipedia, the free encyclopedia

La forza del destino (Italian pronunciation: [la ˈfɔrtsa del deˈstiːno]; *The Power of Fate*,[1] often translated *The Force of Destiny*) is an Italian opera by Giuseppe Verdi. The libretto was written by Francesco Maria Piave based on a Spanish drama, *Don Álvaro o la fuerza del sino* (1835), by Ángel de Saavedra, 3rd Duke of Rivas, with a scene adapted from Friedrich Schiller's *Wallensteins Lager* (*Wallenstein's Camp*). It was first performed in the Bolshoi Kamenny Theatre of Saint Petersburg, Russia, on 29 November 1862 O.S. (N.S. 10 November).

La forza del destino is frequently performed, and there have been a number of complete recordings. In addition, the overture (to the revised version of the opera) is part of the standard repertoire for orchestras, often played as the opening piece at concerts.

Performance history [edit]

Revisions [edit]

After its premiere in Russia, *La forza* underwent some revisions and made its debut abroad with performances in Rome in 1863 under the title *Don Alvaro*. Performances followed in Madrid (with the Duke of Rivas, the play's author, in attendance) and the opera subsequently travelled to New York, Vienna (1865), Buenos Aires (1866), and London (1867).[citation needed]

Following these productions, Verdi made further, more extensive revisions to the opera with

La forza del destino

Opera by Giuseppe Verdi

c. 1870 poster by Charles Lecocq

Librettist Francesco Maria Piave

Type here to search

10°C Rain ENG 10:19 10/04/2023

Playing field

Let (M, g) be a closed Riemannian d-manifold.

Consider a diffeomorphism $\varphi : M \to M$. This is the unknown quantity of elasticity theory.

Second Riemannian metric $h := \varphi^* g$, the pullback of g.

A pair of metrics, g and h, allows us to write down an action (variational functional).

Strain tensor

Linear algebra: a pair of non-degenerate symmetric bilinear forms $g, h : V \times V \to \mathbb{R}$ in a real finite-dimensional vector space V defines an invertible linear operator $L : V \to V$ via the formula

$$h(u, v) = g(Lu, v), \qquad \forall u, v \in V.$$

Convenient to subtract the identity operator,

$$S := L - \mathrm{Id}$$

Definition of strain tensor:

$$S^\alpha{}_\beta(x) := [g^{\alpha\gamma}(x)]\,[h_{\gamma\beta}(x)] - \delta^\alpha{}_\beta\,.$$

Describes, pointwise, linear map in the fibres of the tangent bundle

$$v^\alpha \mapsto S^\alpha{}_\beta\, v^\beta.$$

Scalar invariants of the strain tensor

Obvious choice: $\operatorname{tr}(S^k)$, $k = 1, 2, \ldots, d$.

More convenient choice:

$$e_1(\varphi) := \operatorname{tr} S = \lambda_1 + \lambda_2 + \ldots + \lambda_d \,,$$

$$e_2(\varphi) := \frac{1}{2}\left[(\operatorname{tr} S)^2 - \operatorname{tr}(S^2)\right] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{d-1}\lambda_d \,,$$

$$\vdots$$

$$e_d(\varphi) := \det S = \lambda_1\lambda_2 \ldots \lambda_d \,.$$

Elementary symmetric polynomials. The λ_j are eigenvalues of S.

Action (potential energy of elastic deformation)

$$\int_M \mathcal{L}\big(e_1(\varphi), e_2(\varphi), \ldots, e_d(\varphi)\big)\, \sqrt{\det g}\; dx\,,$$

where $\mathcal{L}$ is some prescribed smooth real-valued function of d real variables and $dx := dx^1\, dx^2\, \ldots\, dx^d$.

Describing diffeomorphism in terms of a vector field

First approach Use integral curves of a vector field. Impossible: J.Milnor 1983.

Second approach Use geodesics.

Connect a point $P \in M$ with the point $\varphi(P) \in M$ by a geodesic $\gamma : [0,1] \to M$, so that $\gamma(0) = P$ and $\gamma(1) = \varphi(P)$. Parameterise the geodesic in such a way that $\gamma(t)$ is a solution of the equation

$$\ddot{\gamma}^{\lambda} + \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \dot{\gamma}^{\mu} \dot{\gamma}^{\nu} = 0,$$

where the dot stands for differentiation in t.

Define the vector field of displacements as

$$u : M \ni P \mapsto \dot{\gamma}(0) \in TM.$$

Linear elasticity

- Linearise the strain tensor with respect to the vector field of displacements u.
- Choose action quadratic in u.

Action reads

$$\frac{1}{2}\int_M \left(\lambda\,(\nabla_\alpha u^\alpha)^2 + \mu\,(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha)\,\nabla^\alpha u^\beta\right)\,\sqrt{\det g}\;dx\,,$$

where λ and μ are Lamé coefficients.

Journal of Mathematical Physics ARTICLE scitation.org/journal/jmp

Spacetime diffeomorphisms as matter fields

Cite as: J. Math. Phys. 61, 111508 (2020); doi: 10.1063/1.5140425
Submitted: 27 November 2019 • Accepted: 31 October 2020 •
Published Online: 17 November 2020

Matteo Capoferri[a] and Dmitri Vassiliev[b]

AFFILIATIONS

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom

[a] **Current address:** School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4AG, United Kingdom.
[b] **Author to whom correspondence should be addressed:** d.vassiliev@ucl.ac.uk

ABSTRACT

We work on a 4-manifold equipped with Lorentzian metric g and consider a volume-preserving diffeomorphism that is the unknown quantity of our mathematical model. The diffeomorphism defines a second Lorentzian metric h, the pullback of g. Motivated by elasticity theory, we introduce a Lagrangian expressed algebraically (without differentiations) via our pair of metrics. Analysis of the resulting nonlinear field equations produces three main results. First, we show that for Ricci-flat manifolds, our linearized field equations are Maxwell's equations in the Lorenz gauge with exact current. Second, for Minkowski space, we construct explicit massless solutions of our nonlinear field equations; these come in two distinct types, right-handed and left-handed. Third, for Minkowski space, we construct explicit massive solutions of our nonlinear field equations; these contain a positive parameter that has the geometric meaning of quantum mechanical mass and a real parameter that may be interpreted as electric charge. In constructing explicit solutions of nonlinear field equations, we resort to group-theoretic ideas: we identify special four-dimensional subgroups of the Poincaré group and seek diffeomorphisms compatible with their action in a suitable sense.

Spectral problem for linear elasticity:

$$-\mu\left(\nabla_\beta\nabla^\beta u^\alpha + \mathrm{Ric}^\alpha{}_\beta u^\beta\right) - (\lambda+\mu)\nabla^\alpha\nabla_\beta u^\beta = \Lambda u^\alpha.$$

Possible boundary conditions.

- Dirichlet.
- Free boundary. This is not the Neumann boundary condition.

Historical overview 1

1885 Lord Rayleigh discovers *Rayleigh wave*. Wave runs along free boundary and exponentially decays towards interior. Let

$$R_\alpha(w) := w^3 - 8w^2 + 8\,(3 - 2\alpha)\,w + 16\,(\alpha - 1)\,,$$

where

$$\alpha := \frac{\mu}{\lambda + 2\mu}.$$

The cubic equation $R_\alpha(w) = 0$ has three roots w_j, $j = 1, 2, 3$, over $\mathbb{C}$, where w_1 is the distinguished real root in the interval $(0, 1)$. Put

$$\gamma_R := \sqrt{w_1}\,.$$

The subscript R in γ_R stands for "Rayleigh". The quantity

$$c_R := \sqrt{\mu}\,\gamma_R$$

has the physical meaning of velocity of Rayleigh's surface wave.

Historical overview 2

1912 Peter Debye writes down one-term asymptotic formula for the eigenvalue counting function

$$\mathcal{N}(\Lambda) = a \operatorname{Vol}_d(M)\, \Lambda^{d/2} + o\left(\Lambda^{d/2}\right) \quad \text{as} \quad \Lambda \to +\infty,$$

where

$$a = \frac{1}{(4\pi)^{d/2}\Gamma\left(1+\frac{d}{2}\right)} \left(\frac{d-1}{\mu^{d/2}} + \frac{1}{(\lambda+2\mu)^{d/2}} \right).$$

1915 Hermann Weyl provides rigorous proof.

Historical overview 3

Search for two-term asymptotic formula

$$\mathcal{N}(\Lambda) = a \operatorname{Vol}_d(M)\,\Lambda^{d/2} + b \operatorname{Vol}_{d-1}(\partial M)\,\Lambda^{(d-1)/2} + o\left(\Lambda^{(d-1)/2}\right) \text{ as } \Lambda \to +\infty$$

Second Weyl coefficient b should depend on boundary conditions.

1950 E. W. Montroll publishes incorrect formulae for second Weyl coefficient. Same incorrect formulae as Genquian Liu in 2021.

1960 Lars Onsager and coauthors publish correct formulae for second Weyl coefficient for $d = 3$.

1997 Safarov and Vassiliev book (only results, without details).

- Onsager's results for $d = 3$ checked and confirmed.
- Formulae for second Weyl coefficient for $d = 2$ written down.

Algorithm for the calculation of second Weyl coefficient

- Fix point $x' \in \partial M$, freeze coefficients in operator and boundary conditions and perform Fourier transform $\partial x' \mapsto i\xi'$ along ∂M. Gives spectral problem for a system of ODEs with constant coefficients on semi-axis $[0, +\infty)$. This 1-dimensional spectral problem depends on $(x', \xi') \in T^*\partial M$ as a parameter.
- Need to calculate the spectral shift function $\text{shift}(x', \xi', \Lambda)$ (regularised trace of spectral projection).
- Use ideas from scattering theory:

$$\text{shift}(x', \xi', \Lambda) := \frac{\varphi(x', \xi', \Lambda)}{2\pi} + N(x', \xi', \Lambda),$$

  where $\varphi(x', \xi', \Lambda)$ is *scattering phase* (phase shift) and $N(x', \xi', \Lambda)$ is the eigenvalue counting function of the 1-dimensional spectral problem.
- $$b = \frac{1}{(2\pi)^{d-1}} \int_{T^*\partial M} \text{shift}(x', \xi', 1)\, dx'\, d\xi'.$$

Spectral shift function for Dirichlet boundary conditions

$$\mathrm{shift}_{\mathrm{Dir}}(\xi', \Lambda) =$$

$$\begin{cases} 0 & \text{for } \Lambda \le \mu\|\xi'\|^2, \\ -\frac{1}{\pi}\arctan\left(\sqrt{\left(1 - \frac{\Lambda}{\lambda+2\mu}\frac{1}{\|\xi'\|^2}\right)\left(\frac{\Lambda}{\mu}\frac{1}{\|\xi'\|^2} - 1\right)}\right) - \frac{d-1}{4} & \\ & \text{for } \mu\|\xi'\|^2 < \Lambda < (\lambda+2\mu)\|\xi'\|^2, \\ -\frac{d}{4} & \text{for } \Lambda > (\lambda+2\mu)\|\xi'\|^2, \end{cases}$$

Spectral shift function for free boundary conditions

$$\mathrm{shift}_{\mathrm{free}}(\xi', \Lambda) =$$

$$\begin{cases} 0 & \text{for } \Lambda < \mu\gamma_R^2\|\xi'\|^2, \\ 1 & \text{for } \mu\gamma_R^2\|\xi'\|^2 < \Lambda < \mu\|\xi'\|^2, \\ \frac{1}{\pi}\arctan\left(\frac{\left(\frac{\Lambda}{\mu}\frac{1}{\|\xi'\|^2}-2\right)^2}{4\sqrt{\left(1-\frac{\Lambda}{\lambda+2\mu}\frac{1}{\|\xi'\|^2}\right)\left(\frac{\Lambda}{\mu}\frac{1}{\|\xi'\|^2}-1\right)}}\right)+\frac{d-1}{4} & \text{for } \mu\|\xi'\|^2 < \Lambda < (\lambda+2\mu)\|\xi'\|^2, \\ \frac{d}{4} & \text{for } \Lambda > (\lambda+2\mu)\|\xi'\|^2. \end{cases}$$

Main result (for α and γ_R see one of previous slides)

$$b_{\mathrm{Dir}} = -\frac{\mu^{\frac{1-d}{2}}}{2^{d+1}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right)}\left(\frac{4(d-1)}{\pi}\int_{\sqrt{\alpha}}^{1}\tau^{d-2}\arctan\left(\sqrt{\left(1-\alpha\tau^{-2}\right)\left(\tau^{-2}-1\right)}\right)d\tau + \alpha^{\frac{d-1}{2}} + d - 1\right),$$

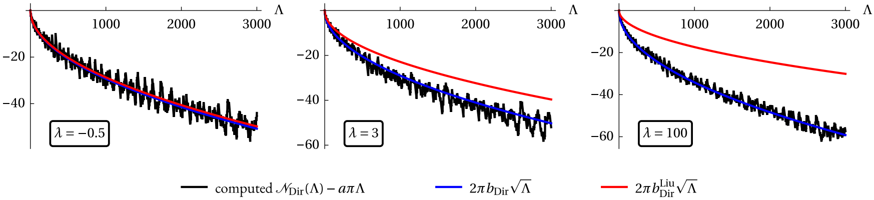
$$b_{\mathrm{free}} = \frac{\mu^{\frac{1-d}{2}}}{2^{d+1}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right)}\left(\frac{4(d-1)}{\pi}\int_{\sqrt{\alpha}}^{1}\tau^{d-2}\arctan\left(\frac{\left(\tau^{-2}-2\right)^{2}}{4\sqrt{\left(1-\alpha\tau^{-2}\right)\left(\tau^{-2}-1\right)}}\right)d\tau + \alpha^{\frac{d-1}{2}} + d - 5 + 4\,\gamma_R^{1-d}\right).$$

d	b_{Dir}
3	$-\dfrac{(2\alpha^2+\alpha+3)\mu}{16\pi(\alpha+1)}$
5	$-\dfrac{(7\alpha^4+12\alpha^3+6\alpha^2+36\alpha+19)\mu^2}{512\pi^2(\alpha+1)^2}$
7	$-\dfrac{(13\alpha^6+36\alpha^5+27\alpha^4+16\alpha^3+147\alpha^2+156\alpha+53)\mu^3}{12288\pi^3(\alpha+1)^3}$
9	$-\dfrac{(99\alpha^8+376\alpha^7+500\alpha^6+200\alpha^5+146\alpha^4+1992\alpha^3+3188\alpha^2+2168\alpha+547)\mu^4}{1572864\pi^4(\alpha+1)^4}$

Table 1: The coefficient b_{Dir} for odd dimensions.

d	b_{free}
3	$\frac{(2\alpha^2-3\alpha+3)\mu}{16\pi(1-\alpha)}$
5	$\frac{(-7\alpha^4+12\alpha^3+14\alpha^2-36\alpha+21)\mu^2}{512\pi^2(1-\alpha)^2}$
7	$\frac{(13\alpha^6-36\alpha^5+30\alpha^4-56\alpha^3+159\alpha^2-168\alpha+62)\mu^3}{12288\pi^3(1-\alpha)^3}$
9	$\frac{(-99\alpha^8+376\alpha^7-516\alpha^6+296\alpha^5+470\alpha^4-2200\alpha^3+3468\alpha^2-2440\alpha+661)\mu^4}{1572864\pi^4(1-\alpha)^4}$

Table 2: The coefficient b_{free} for odd dimensions.


Λ
1000
2000
3000
−20
−40
−60
λ = −0.5
λ = 3
λ = 100
computed 𝒩Dir(Λ) − aπΛ
2πbDir√Λ
2πb^Liu_Dir√Λ