Spectral theory of differential operators: what's it all about and what is its use Part I

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Basic example of a problem in my subject area

Acoustic resonator. Suppose we are studying the vibrations of air

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x_1^2} - \frac{\partial^2 p}{\partial x_2^2} - \frac{\partial^2 p}{\partial x_3^2} = 0$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ subject to boundary conditions

$$\left.\frac{\partial p}{\partial n}\right|_{\partial\Omega} = 0.$$

Here p is the pressure and c is the speed of sound.

Seek solutions in the form $p(x_1, x_2, x_3, t) = p(x_1, x_2, x_3)e^{-i\omega t}$, where ω is the unknown natural frequency.

This leads to an eigenvalue problem:

$$-\Delta p = \lambda p \quad \text{in} \quad \Omega, \qquad \partial p/\partial n|_{\partial\Omega} = 0,$$

where Δ is the Laplacian and $\lambda := \frac{\omega^2}{c^2}$ is the spectral parameter.

Finding eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ is difficult, so one introduces the counting function

$$N(\lambda) := \sum_{0 \leq \lambda_k < \lambda} 1$$

("number of eigenvalues below a given λ") and studies the asymptotic behaviour of $N(\lambda)$ as $\lambda \to +\infty$.

Rayleigh–Jeans law (1905)

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + o(\lambda^{3/2}) \quad \text{as} \quad \lambda \to +\infty,$$

where V is the volume of the resonator.

Rayleigh's "proof" of the Rayleigh–Jeans law

Suppose Ω is a cube with side length a. Then the eigenvalues and eigenfunctions can be calculated explicitly:

$$\psi_{\mathbf{k}} = \cos\left(\frac{\pi k_1 x_1}{a}\right)\cos\left(\frac{\pi k_2 x_2}{a}\right)\cos\left(\frac{\pi k_3 x_3}{a}\right),$$

$$\lambda_{\mathbf{k}} = \frac{\pi^2}{a^2}\|\mathbf{k}\|^2 = \frac{\pi^2}{a^2}(k_1^2 + k_2^2 + k_3^2),$$

where $\mathbf{k} = (k_1, k_2, k_3)$ and k_1, k_2, k_3 are nonnegative integers. $N(\lambda)$ is the number of integer lattice points in the nonnegative octant of a ball of radius $\frac{a}{\pi}\sqrt{\lambda}$, so

$$N(\lambda) \approx \frac{1}{8}\left(\frac{4}{3}\pi\left(\frac{a}{\pi}\sqrt{\lambda}\right)^3\right) = \frac{a^3}{6\pi^2}\lambda^{3/2} = \frac{V}{6\pi^2}\lambda^{3/2}.$$

Jeans' contribution to the Rayleigh–Jeans law

"It seems to me that Lord Rayleigh has introduced an unnecessary factor 8 by counting negative as well as positive values of his integers".

1910: Lorentz visits Göttingen at Hilbert's invitation and delivers a series of lectures "Old and new problems in physics". Lorentz states the Rayleigh–Jeans law as a mathematical conjecture. Hermann Weyl is in the audience.

1912: Weyl publishes a rigorous proof of Rayleigh–Jeans law. Almost incomprehensible.

Comprehensible proof: in R.Courant and D.Hilbert, *Methods of Mathematical Physics* (1924).

Courant's method

Approximate domain Ω by a collection of small cubes, setting Dirichlet or Neumann boundary conditions on boundaries of cubes. Setting extra Dirichlet conditions raises the eigenvalues whereas setting extra Neumann conditions lowers the eigenvalues.

Remains only to

- choose size of cubes correctly (in relation to λ) and
- estimate contribution of bits of domain near the boundary (we throw them out).

Weyl's Conjecture (1913)

One can do better and prove two-term asymptotic formulae for the counting function. Say, for the case of the Laplacian in 3D with Neumann boundary conditions Weyl's Conjecture reads

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + \frac{S}{16\pi}\lambda + o(\lambda) \quad \text{as} \quad \lambda \to +\infty,$$

where S is the surface area of ∂M.

For Dirichlet boundary conditions change sign in second term.

Similar conjecture in arbitrary dimension d.

Search for a general method

Winner: Boris Levitan's wave equation method.

Instead of studying the spectral problem

$$-\Delta u = \lambda u$$

study the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0.$$

Then perform inverse Fourier transform, from the variable t (time) to the variable $\sqrt{\lambda}$ (frequency).

Fourier Tauberian theorems allow us to perform the inverse Fourier transform using incomplete information, with control of error terms. Similar to Tauberian theorems in analytic number theory.

Timeline leading up to Ivrii's proof of the Weyl conjecture

1952 B.M.Levitan

1968 L.Hörmander

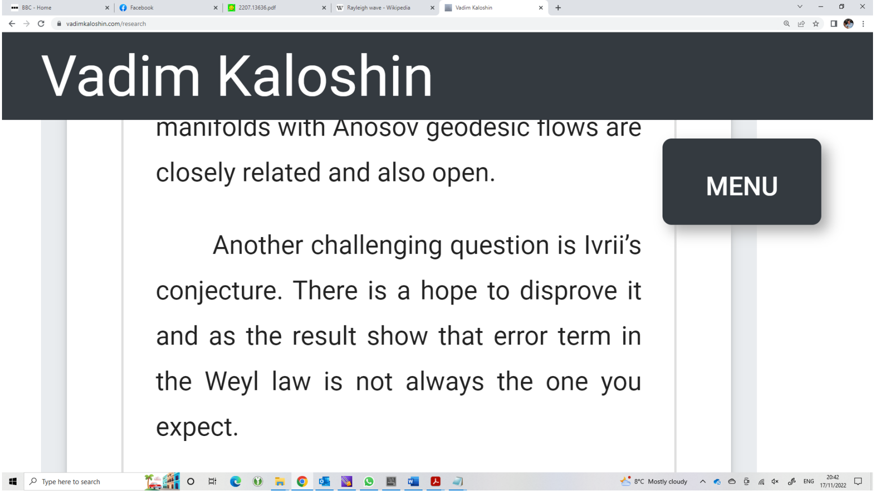
1975 J.J.Duistermaat and V.W.Guillemin

1980 V.Ivrii

Theorem (Victor Ivrii) Weyl's conjecture holds if periodic billiard trajectories have measure zero.

Condition on periodic billiard trajectories is required to exclude situations with too many symmetries. Already noticed by Duistermaat and Guillemin.

Conjecture (Victor Ivrii) In the Euclidean case periodic billiard trajectories have measure zero.



Vadim Kaloshin

manifolds with Anosov geodesic flows are closely related and also open.

MENU

Another challenging question is Ivrii's conjecture. There is a hope to disprove it and as the result show that error term in the Weyl law is not always the one you expect.

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Mathematics > Dynamical Systems

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Absolutely Periodic Billiard Orbits of Arbitrarily High Order

Keagan G. Callis

We show that for any natural number n, the set of domains containing absolutely periodic orbits of order n are dense in the set of bounded strictly convex domains with smooth boundary. The proof that such an orbit exists is an extension to billiard maps of the results of a paper by Gonchenko, Shilnikov, and Turaev, where it is proved that such maps are dense in Newhouse domains in regions of real-analytic area-preserving two-dimensional maps. Our result is a step toward disproving a conjecture that no absolutely periodic billiard orbits of infinite order exist in Euclidean billiards and is also an indication that Ivrii's Conjecture about the measure set of periodic orbits may not be true.

Subjects: **Dynamical Systems (math.DS)**

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