Geometric wave propagator on Riemannian manifolds

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Playing field

Let (M, g) be a connected closed Riemannian manifold of dimension $d \geq 2$. Local coordinates x^α, $\alpha = 1, \dots, d$.

Will work with scalar functions $u : M \to \mathbb{C}$.

Inner product

$$(u, v) := \int_M \overline{u(x)}\, v(x)\, \rho(x)\, dx\,,$$

where $\rho(x) := \sqrt{\det g_{\mu\nu}(x)}$ and $dx = dx^1 \dots dx^d$.

Laplace–Beltrami operator

$$\Delta := \rho(x)^{-1} \frac{\partial}{\partial x^\mu} \rho(x) \, g^{\mu\nu}(x) \, \frac{\partial}{\partial x^\nu} \, .$$

Eigenvalues and normalised eigenfunctions:

$$-\Delta v_k = \lambda_k^2 v_k \, ,$$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \to +\infty.$$

The propagator

Time-dependent unitary operator

$$U(t) := e^{-it\sqrt{-\Delta}} = \sum_{k=0}^{\infty} \int_M e^{-it\lambda_k} \, v_k(x) \, \overline{v_k(y)} \, (\,\cdot\,) \, \rho(y) \, dy \, .$$

Propagator is the solution of the operator-valued Cauchy problem

$$U(0) = \mathrm{Id}$$

for the half-wave equation

$$\left(-i \, \frac{\partial}{\partial t} + \sqrt{-\Delta}\right) U(t) = 0 \, .$$

Task

Construct the propagator $U(t)$ *approximately* and do it *explicitly*.

'Approximately' means 'modulo an integral operator with infinitely smooth integral kernel'. 'Explicitly' means 'by solving ODEs as opposed to PDEs'.

Levitan, Hörmander and many others: can be done using microlocal techniques. But issues with the classical construction.

- It is not invariant under changes of local coordinates.
- It is local in space.
- It is local in time.

Locality in time is especially serious (topological obstructions):

$$U(t) = U(t - t_j) \circ U(t_j - t_{j-1}) \circ \cdots \circ U(t_2 - t_1) \circ U(t_1).$$

The basics of microlocal analysis

Consider $M=\mathbb{R}^d$ equipped with Cartesian coordinates and metric $g_{\alpha\beta}(x)=\delta_{\alpha\beta}$. The exact explicit formula for the propagator reads

$$U(t)=\frac{1}{(2\pi)^d}\int e^{i\varphi(t,x;y,\eta)}\,(\,\cdot\,)\,dy\,d\eta\,,$$

where

$$\varphi:\mathbb{R}\times M\times T^*M\to\mathbb{R},\qquad \varphi(t,x;y,\eta)=(x-y)^\alpha\eta_\alpha-t\|\eta\|$$

is the *phase function* and $d\eta=d\eta_1\ldots d\eta_d$.

In curved space one has to replace the above function φ by a different phase function which feels the geometry of the particular Riemannian manifold (M,g). Microlocal analysis is, in essence, the art of doing the Fourier transform in curved space.

Global invariant construction

Main idea: replace real-valued phase function by complex-valued.

Illustration in $\mathbb{R}^d$: instead of expanding over a trigonometric basis

$$e^{ix^\alpha \eta_\alpha}$$

expand over the set

$$e^{ix^\alpha \eta_\alpha - \frac{\epsilon}{2}\|\eta\|\,\|x\|^2},$$

where $\epsilon > 0$ is a parameter.

Keywords: Gaussian beam, wavelet.

Our construction originates from:

A. Laptev, Yu. Safarov and D. Vassiliev, On global representation of Lagrangian distributions and solutions of hyperbolic equations, *Comm. Pure Appl. Math.* **47** 11 (1994) 1411–1456.

Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*. Amer. Math. Soc., Providence (RI), 1997.

Current talk based on:

M. Capoferri, M. Levitin and D. Vassiliev, *Geometric wave propagator on Riemannian manifolds*, arXiv:1902.06982. To appear in *Comm. Anal. Geom.*

Hamiltonian flow on T^*M

$$h(x,\xi) := \sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\,\xi_\beta} = \|\xi\|\,,$$

$$\begin{cases} \dot{x}^* = h_\xi(x^*,\xi^*)\,, \\ \dot{\xi}^* = -h_x(x^*,\xi^*)\,, \end{cases}$$

$$(x^*,\xi^*)|_{t=0} = (y,\eta)\,.$$

Hamiltonian trajectories

$$(x^*(t;y,\eta),\xi^*(t;y,\eta))$$

play the role of a skeleton in our construction.

The real-valued Levi-Civita phase function

For x close to $x^*(t; y, \eta)$

$$\varphi(t, x; y, \eta) := \int_\gamma \zeta \, dz \,,$$

where the path of integration γ is the (unique) shortest geodesic connecting $x^*(t; y, \eta)$ to x and ζ is the result of the parallel transport of $\xi^*(t; y, \eta)$ along γ.

Nice properties: $(\Delta\varphi)|_{x=x^*} = 0 \,, \; (\varphi_{tt})|_{x=x^*} = 0 \,.$

Problem: it may happen that for some t

$$\det \varphi_{x^\alpha \eta_\beta}\big|_{x=x^*} = 0 \,.$$

The complex-valued Levi-Civita phase function

For x close to $x^*(t; y, \eta)$

$$\varphi(t, x; y, \eta) := \int_\gamma \zeta \, dz + \frac{i\,\epsilon}{2} h(y, \eta) \, \mathrm{dist}^2(x, x^*(t; y, \eta)) \,,$$

where $\epsilon > 0$ is a parameter.

Fact: we are now guaranteed to have

$$\det \varphi_{x^\alpha \eta_\beta}\Big|_{x=x^*} \neq 0 \,.$$

The global invariant formula for the propagator reads

$$U(t) \overset{\operatorname{mod} C^\infty}{=} \frac{1}{(2\pi)^n} \int_{T^*M} e^{i\varphi(t,x;y,\eta)}\, \mathfrak{a}(t;y,\eta)\, \chi(t,x;y,\eta)\, w(t,x;y,\eta)\, (\,\cdot\,)\, dy\, d\eta$$

where

- the scalar function $\mathfrak{a} : \mathbb{R} \times (T^*M \setminus \{0\}) \to \mathbb{C}$ is the global invariantly defined symbol,
- χ is a cut-off and
- w is a weight defined as

$$w(t,x;y,\eta) := [\rho(x)]^{-1/2}\, [\rho(y)]^{1/2} \left[\det{}^2\left(\varphi_{x^\alpha \eta_\beta}(t,x;y,\eta)\right)\right]^{1/4}.$$

The oscillatory integral is completely determined by its symbol.

Concept of *quantization*.

Calculating the symbol of the propagator

Expansion into components $\mathfrak{a}_{-k}$ positively homogeneous in momentum η of degree $-k$,

$$\mathfrak{a}(t;y,\eta) \sim \sum_{k=0}^{\infty} \mathfrak{a}_{-k}(t;y,\eta)\,.$$

The $\mathfrak{a}_{-k}$ are the unknowns of our construction.

We call the function $\mathfrak{a}_0(t;y,\eta)$ the *principal symbol of the propagator*. It turns out that $\mathfrak{a}_0(t;y,\eta)=1$.

We call the function $\mathfrak{a}_{-1}(t;y,\eta)$ the *subprincipal symbol of the propagator*.

We have

- an explicit formula for the subprincipal symbol $\mathfrak{a}_{-1}(t;y,\eta)$ and
- an algorithm for the calculation of $\mathfrak{a}_{-k}(t;y,\eta)$, $k=2,3,\ldots$.

Subprincipal symbol of the propagator on the 2-sphere

For general $\epsilon > 0$

$$\mathfrak{a}_{-1}(t; y, \eta) = \frac{i\,t}{8\,\|\eta\|} + \frac{i\sin(2t) - 4\epsilon\sin^2(t) + 3\,i\,\epsilon^2\sin(2t) + 6\epsilon^3\sin^2(t)}{48\,\|\eta\|\,(\cos(t) - i\epsilon\sin(t))^2}\,.$$

If we take $\epsilon = 1$ formula simplifies and reads

$$\mathfrak{a}_{-1}(t; y, \eta) = \frac{i\,t}{8\,\|\eta\|} + \frac{2e^{2\,i\,t} + 3e^{4\,i\,t} - 5}{96\,\|\eta\|}\,.$$

For $\epsilon = 0$ formula becomes

$$\mathfrak{a}_{-1}(t; y, \eta) = \frac{i}{24\,\|\eta\|}\,(3t + \tan(t))\,.$$

Subprincipal symbol for the hyperbolic plane

Setting $\epsilon = 0$ we get

$$\mathfrak{a}_{-1}(t; y, \eta) = -\frac{i}{24\,\|\eta\|}\left(3t + \tanh(t)\right).$$

Small time expansion of subprincipal symbol of propagator

Let $\epsilon = 0$. Then

$$\mathfrak{a}_{-1}(t; y, \eta) = \frac{i}{12\,\|\eta\|}\,\mathcal{R}(y)\,t + O(t^2) \quad \text{as} \quad t \to 0\,,$$

where $\mathcal{R}$ is scalar curvature.

This allows us to recover the *third* Weyl coefficient in the asymptotic expansion of the local counting function.

Weyl coefficients

Local counting function: $N(y,\lambda) := \sum_{\lambda_k<\lambda} |v_k(y)|^2\,.$

Global counting function: $N(\lambda) := \sum_{\lambda_k<\lambda} 1 = \int_M N(y;\lambda)\,\rho(y)\,dy\,.$

Mollification:

$$(N' * \mu)(y,\lambda) = c_{d-1}(y)\,\lambda^{d-1} + c_{d-2}(y)\,\lambda^{d-2} + c_{d-3}(y)\,\lambda^{d-3} + \dots$$

as $\lambda \to +\infty$. Here $\mu(\lambda)$ is a mollifier. Turns out that

$$c_{d-1}(y) = \frac{S_{d-1}}{(2\pi)^d}\,, \quad c_{d-2}(y) = 0\,, \quad c_{d-3}(y) = \frac{d-2}{12}\,\mathcal{R}(y)\,c_{d-1}(y)\,,$$

where S_{d-1} is the volume of the $(d-1)$-dimensional unit sphere.

Maslov index

Calculate the increment of

$$-\frac{1}{2\pi} \arg \det{}^2 \varphi_{x^\alpha \eta_\beta}\Big|_{x=x^*}$$

along the closed geodesic.

For the 2-sphere

$$\det \varphi_{x^\alpha \eta_\beta}\Big|_{x=x^*} = [\rho(x)]\,[\rho(y)]^{-1}\,[\cos(t) - i\,\epsilon\,\sin(t)]\,,$$

which gives Maslov index 2.

Other approaches to global construction of propagator

Publications by applied mathematicians and physicists working in solid state physics and electromagnetic wave propagation. Inspired by geometric optics and use concept of Gaussian beam.

Publications by James Ralston and his students, 1983 to date. Also inspired by geometric optics and use concept of Gaussian beam.

A. Melin and J. Sjöstrand, 1976. Real analytic manifold, complexification of phase space.

Melin and Sjöstrand's techniques later adopted and developed by S. Zelditch, 2007 and 2014.

Adapting our method to other meaningful problems

1 Massless Dirac operator on an oriented Riemannian 3-manifold.

M. Capoferri and D. Vassiliev, *Global propagator for the massless Dirac operator and spectral asymptotics*, arXiv:2004.06351.

2 Operator $\operatorname{curl} := *d$ on an oriented Riemannian 3-manifold.

3 Extension of our method to the Lorentzian setting. Start with the (scalar) wave equation, then massless Dirac, then Maxwell ...

M. Capoferri, C. Dappiaggi and N. Drago, *Global wave parametrices on globally hyperbolic spacetimes*, arXiv:2001.04164.

4 Linearised Einstein equations?