Global hyperbolic propagators in curved space

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The scalar wave propagator

Let (M, g) be a connected closed Riemannian manifold and let Δ be the corresponding Laplace–Beltrami operator. The operator

$$U(t) := e^{-it\sqrt{-\Delta}}$$

is called the *scalar wave propagator*.

It is the (distributional) solution of the Cauchy problem

$$U(0) = \mathrm{Id}$$

for the pseudodifferential hyperbolic half-wave equation

$$\left(-i\,\frac{\partial}{\partial t} + \sqrt{-\Delta}\right) U = 0\,.$$

Microlocal techniques (Levitan, Hörmander): one can construct the propagator $U(t)$ *approximately*, modulo an integral operator with smooth integral kernel.

Here $U(t)$ is written locally, in time and in space, as a composition of oscillatory integrals whose amplitudes and phase functions are obtained by solving ODEs, as opposed to PDEs.

Issues with the classical construction.

- It is not invariant under changes of local coordinates.
- It is local in space.
- It is local in time.

Locality in time is especially serious (topological obstructions):

$$U(t) = U(t - t_j) \circ U(t_j - t_{j-1}) \circ \cdots \circ U(t_2 - t_1) \circ U(t_1).$$

Global invariant construction

Basic idea: replace real-valued phase function by complex-valued.

Illustration in $\mathbb{R}^n$: instead of expanding over a trigonometric basis

$$e^{ix^\alpha \xi_\alpha}$$

expand over the set

$$e^{ix^\alpha \xi_\alpha - \frac{\epsilon}{2}\|\xi\|\,\|x\|^2},$$

where $\epsilon > 0$ is a parameter. Here x^α, $\alpha = 1, \ldots, n$, are Cartesian coordinates and ξ_α, $\alpha = 1, \ldots, n$, is momentum.

Keywords: Gaussian beam, wavelet.

Our construction originates from:

A. Laptev, Yu. Safarov and D. Vassiliev, On global representation of Lagrangian distributions and solutions of hyperbolic equations, *Comm. Pure Appl. Math.* **47** 11 (1994) 1411–1456.

Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*. Amer. Math. Soc., Providence (RI), 1997, 1998.

M. Capoferri, M. Levitin and D. Vassiliev, *Geometric wave propagator on Riemannian manifolds*, arXiv:1902.06982.

Hamiltonian flow on T^*M

$$h(x,\xi) := \sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\,\xi_\beta} = \|\xi\|\,,$$

$$\begin{cases} \dot{x}^* = h_\xi(x^*,\xi^*)\,, \\ \dot{\xi}^* = -h_x(x^*,\xi^*)\,, \end{cases}$$

$$(x^*,\xi^*)|_{t=0} = (y,\eta)\,.$$

Hamiltonian trajectories

$$(x^*(t;y,\eta),\xi^*(t;y,\eta))$$

play the role of a skeleton in our construction.

The real-valued Levi-Civita phase function

For x close to $x^*(t; y, \eta)$

$$\varphi(t, x; y, \eta) := \int_\gamma \zeta \, dz \,,$$

where the path of integration γ is the (unique) shortest geodesic connecting $x^*(t; y, \eta)$ to x and ζ is the result of the parallel transport of $\xi^*(t; y, \eta)$ along γ.

Problem: it may happen that

$$\det \varphi_{x^\alpha \eta_\beta}\Big|_{x=x^*} = 0 \,.$$

The complex-valued Levi-Civita phase function

For x close to $x^*(t; y, \eta)$

$$\varphi(t, x; y, \eta) := \int_\gamma \zeta \, dz + \frac{i\,\epsilon}{2} \, h(y, \eta) \, \mathrm{dist}^2(x, x^*(t; y, \eta)) \,,$$

where $\epsilon > 0$ is a parameter.

Fact: we are now guaranteed to have

$$\det \varphi_{x^\alpha \eta_\beta} \big|_{x = x^*} \neq 0 \,.$$

The global invariant formula for the propagator reads

$$U(t) \overset{\mathrm{mod}\ C^\infty}{=} \frac{1}{(2\pi)^n} \int_{T^*M} e^{i\varphi(t,x;y,\eta)}\, \mathfrak{a}(t;y,\eta)\, \chi(t,x;y,\eta)\, w(t,x;y,\eta)\, (\,\cdot\,)\, \mathrm{d}y\, \mathrm{d}\eta$$

where

- the scalar function $\mathfrak{a} : \mathbb{R} \times (T^*M \setminus \{0\}) \to \mathbb{C}$ is the global invariantly defined symbol,
- χ is a cut-off,
- w is a weight defined as

$$w(t,x;y,\eta) := [\rho(x)]^{-1/2}\, [\rho(y)]^{1/2} \left[\det{}^2\left(\varphi_{x^\alpha \eta_\beta}(t,x;y,\eta)\right)\right]^{1/4},$$

- ρ is the Riemannian density and
- $\mathrm{d}y = \mathrm{d}y^1 \ldots \mathrm{d}y^n$, $\mathrm{d}\eta = \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n$.

The oscillatory integral is completely determined by its symbol.

Concept of *quantization*.

Calculating the symbol of the propagator

Expansion into components $\mathfrak{a}_{-k}$ positively homogeneous in momentum η of degree $-k$,

$$\mathfrak{a}(t;y,\eta) \sim \sum_{k=0}^{\infty} \mathfrak{a}_{-k}(t;y,\eta)\,.$$

The $\mathfrak{a}_{-k}$ are the unknowns of our construction.

We call the function $\mathfrak{a}_0(t;y,\eta)$ the *principal symbol of the propagator*. It turns out that $\mathfrak{a}_0(t;y,\eta)=1$.

We call the function $\mathfrak{a}_{-1}(t;y,\eta)$ the *subprincipal symbol of the propagator*.

We have

- an explicit formula for the subprincipal symbol $\mathfrak{a}_{-1}(t;y,\eta)$ and
- an algorithm for the calculation of $\mathfrak{a}_{-k}(t;y,\eta)$, $k=2,3,\ldots$.

Subprincipal symbol of the propagator on a 2-sphere

For general $\epsilon > 0$

$$\mathfrak{a}_{-1}(t;y,\eta) = \frac{i\,t}{8\,\|\eta\|} + \frac{i\sin(2t) - 4\epsilon\sin^2(t) + 3\,i\,\epsilon^2\sin(2t) + 6\epsilon^3\sin^2(t)}{48\,\|\eta\|\,(\cos(t) - i\epsilon\sin(t))^2}\,.$$

If we take $\epsilon = 1$ formula simplifies and reads

$$\mathfrak{a}_{-1}(t;y,\eta) = \frac{i\,t}{8\,\|\eta\|} + \frac{2e^{2\,i\,t} + 3e^{4\,i\,t} - 5}{96\,\|\eta\|}\,.$$

For $\epsilon = 0$ formula becomes

$$\mathfrak{a}_{-1}(t;y,\eta) = \frac{i}{24\,\|\eta\|}\,(3t + \tan(t))\,.$$

Subprincipal symbol for the hyperbolic plane

Setting $\epsilon = 0$ we get

$$\mathfrak{a}_{-1}(t; y, \eta) = -\frac{i}{24\,\|\eta\|}\left(3t + \tanh(t)\right).$$

Small time expansion of subprincipal symbol of propagator

Theorem (Capoferri–Levitin–Vassiliev) Let $\epsilon = 0$. Then

$$\mathfrak{a}_{-1}(t; y, \eta) = \frac{i}{12\,\|\eta\|}\,\mathcal{R}(y)\,t + O(t^2) \quad \text{as} \quad t \to 0\,,$$

where $\mathcal{R}$ is scalar curvature.

This theorem allows us to recover the third Weyl coefficient in the asymptotic expansion of the local counting function:

$$N(y, \lambda) := \sum_{\lambda_k < \lambda} |v_k(y)|^2\,,$$

$$(N' * \mu)(y, \lambda) = a_{d-1}(y)\,\lambda^{d-1} + a_{d-2}(y)\,\lambda^{d-2} + a_{d-3}(y)\,\lambda^{d-3} + \dots$$

as $\lambda \to +\infty$. Here $\mu(\lambda)$ is a mollifier.

Maslov index

Calculate the increment of

$$-\frac{1}{2\pi} \arg \det{}^2 \varphi_{x^\alpha \eta_\beta}\Big|_{x=x^*}$$

along the closed geodesic.

For the 2-sphere

$$\det \varphi_{x^\alpha \eta_\beta}\Big|_{x=x^*} = [\rho(x)]\,[\rho(y)]^{-1}\,[\cos(t) - i\,\epsilon\,\sin(t)]\,,$$

which gives Maslov index 2.

Work in progress

1 Massless Dirac operator on an oriented Riemannian 3-manifold.

2 Operator $\operatorname{curl} := *d$ on an oriented Riemannian 3-manifold.

Elementary particles come in two types — fermions and bosons.

The most basic fermion is the massless neutrino, whereas the most basic boson is the photon. The massless neutrino is described by the massless Dirac operator, whereas the photon is described by Maxwell's equations. The latter reduce to

$$\left(-i\,\frac{\partial}{\partial t}+\operatorname{curl}\right)u=0\,,$$

$$\delta u=0,$$

where u is the unknown complex-valued 1-form.

General challenges for massless Dirac and for curl

1 We are now dealing with systems.

2 The spectrum is no longer semi-bounded.

3 We now need *two* oscillatory integrals for our propagators. These correspond to positive and negative eigenvalues.

4 The spectrum is asymmetric about zero. One has to deal with positive and negative eigenvalues separately by introducing two local counting functions

$$N_\pm(y,\lambda) := \sum_{0<\pm\lambda_k<\lambda} \|v_k(y)\|^2 \,.$$

One expects to observe asymmetry in the *sixth* Weyl coefficient.

Challenge specific to the massless Dirac operator

The massless Dirac operator W is a 2×2 matrix first order differential operator. It is uniquely defined by metric and spin structure modulo the gauge transformation

$$W \mapsto G^* W G,$$

where

$$G : M \to \mathrm{SU}(2)$$

is an arbitrary smooth special unitary matrix-function.

We need to construct the massless Dirac propagator

$$U(t) := e^{-itW},$$

in a geometrically meaningful fashion, in the sense that it respects the above gauge transformations.

Challenges specific to the operator curl

1 The operator curl is not elliptic: the determinant of its principal symbol is identically zero.

2 One can turn curl into an elliptic operator by switching to the 'extended curl' operator $\mathrm{Curl} := \begin{pmatrix} \mathrm{curl} & d \\ \delta & 0 \end{pmatrix}$ acting in the space $\Omega^1(M) \oplus \Omega^0(M)$ and constructing its propagator $e^{-it\,\mathrm{Curl}}$.

3 The principal symbol of the 'extended curl' operator Curl has double eigenvalues. Hence, the hyperbolic system

$$\left(-i\,\frac{\partial}{\partial t} + \mathrm{Curl}\right) v = 0$$

has double characteristics.

4 The Schwartz kernel of the curl propagator is a two-point tensor.