Spectral theory of differential operators: what's it all about and what is its use

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Basic example of a problem in my subject area

Acoustic resonator. Suppose we are studying the vibrations of air

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x_1^2} - \frac{\partial^2 p}{\partial x_2^2} - \frac{\partial^2 p}{\partial x_3^2} = 0$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ subject to boundary conditions

$$\left.\frac{\partial p}{\partial n}\right|_{\partial\Omega} = 0.$$

Here p is the pressure and c is the speed of sound.

Seek solutions in the form $p(x_1, x_2, x_3, t) = p(x_1, x_2, x_3)e^{-i\omega t}$, where ω is the unknown natural frequency.

This leads to an eigenvalue problem:

$$-\Delta p = \lambda p \quad \text{in} \quad \Omega, \qquad \partial p/\partial n|_{\partial\Omega} = 0,$$

where Δ is the Laplacian and $\lambda := \frac{\omega^2}{c^2}$ is the spectral parameter.

Finding eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ is difficult, so one introduces the counting function

$$N(\lambda) := \sum_{0 \leq \lambda_k < \lambda} 1$$

("number of eigenvalues below a given λ") and studies the asymptotic behaviour of $N(\lambda)$ as $\lambda \to +\infty$.

Rayleigh–Jeans law (1905)

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + o(\lambda^{3/2}) \quad \text{as} \quad \lambda \to +\infty,$$

where V is the volume of the resonator.

Rayleigh's "proof" of the Rayleigh–Jeans law

Suppose Ω is a cube with side length a. Then the eigenvalues and eigenfunctions can be calculated explicitly:

$$\psi_{\mathbf{k}} = \cos\left(\frac{\pi k_1 x_1}{a}\right)\cos\left(\frac{\pi k_2 x_2}{a}\right)\cos\left(\frac{\pi k_3 x_3}{a}\right),$$

$$\lambda_{\mathbf{k}} = \frac{\pi^2}{a^2}\|\mathbf{k}\|^2 = \frac{\pi^2}{a^2}(k_1^2 + k_2^2 + k_3^2),$$

where $\mathbf{k} = (k_1, k_2, k_3)$ and k_1, k_2, k_3 are nonnegative integers. $N(\lambda)$ is the number of integer lattice points in the nonnegative octant of a ball of radius $\frac{a}{\pi}\sqrt{\lambda}$, so

$$N(\lambda) \approx \frac{1}{8}\left(\frac{4}{3}\pi\left(\frac{a}{\pi}\sqrt{\lambda}\right)^3\right) = \frac{a^3}{6\pi^2}\lambda^{3/2} = \frac{V}{6\pi^2}\lambda^{3/2}.$$

Jeans' contribution to the Rayleigh–Jeans law

"It seems to me that Lord Rayleigh has introduced an unnecessary factor 8 by counting negative as well as positive values of his integers".

1910: Lorentz visits Göttingen at Hilbert's invitation and delivers a series of lectures "Old and new problems in physics". Lorentz states the Rayleigh–Jeans law as a mathematical conjecture. Hermann Weyl is in the audience.

1912: Weyl publishes a rigorous proof of Rayleigh–Jeans law. Almost incomprehensible.

Comprehensible proof: in R.Courant and D.Hilbert, *Methods of Mathematical Physics* (1924).

Courant's method

Approximate domain Ω by a collection of small cubes, setting Dirichlet or Neumann boundary conditions on boundaries of cubes. Setting extra Dirichlet conditions raises the eigenvalues whereas setting extra Neumann conditions lowers the eigenvalues.

Remains only to

- choose size of cubes correctly (in relation to λ) and
- estimate contribution of bits of domain near the boundary (we throw them out).

General statement of the problem

Let M be a compact n-dimensional manifold with boundary ∂M. Consider the spectral problem for an elliptic self-adjoint semi-bounded from below differential operator of even order $2m$:

$$Au = \lambda u \quad \text{on} \quad M, \qquad (B^{(j)}u)\Big|_{\partial M} = 0, \quad j = 1, \ldots, m.$$

Has been proven (by many authors over many years) that

$$N(\lambda) = a\lambda^{n/(2m)} + o(\lambda^{n/(2m)}) \quad \text{as} \quad \lambda \to +\infty$$

where the constant a is written down explicitly.

Weyl's Conjecture (1913)

One can do better and prove two-term asymptotic formulae for the counting function. Say, for the case of the Laplacian in 3D with Neumann boundary conditions Weyl's Conjecture reads

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + \frac{S}{16\pi}\lambda + o(\lambda) \quad \text{as} \quad \lambda \to +\infty,$$

where S is the surface area of ∂M. For a general partial differential operator of order $2m$ Weyl's Conjecture reads

$$N(\lambda) = a\lambda^{n/(2m)} + b\lambda^{(n-1)/(2m)} + o(\lambda^{(n-1)/(2m)}) \quad \text{as} \quad \lambda \to +\infty$$

where the constant b can also be written down explicitly.

For the case of a second order operator Weyl's Conjecture was proved by V.Ivrii in 1980.

I proved it for operators of arbitrary order in 1984.

My main research publication:

Yu.Safarov and D.Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, American Mathematical Society, 1997 (hardcover), 1998 (softcover).

"In the reviewer's opinion, this book is indispensable for serious students of spectral asymptotics". Lars Hörmander for the Bulletin of the London Mathematical Society.

Two basic issues

- Prove the existence of a two-term asymptotics expansion. Need to exclude situations with too many symmetries. Requires the examination of a particular dynamical system, a Hamiltonian billiards on the cotangent bundle. There shouldn't be too many periodic billiard trajectories.
- Derive explicit formula for the second Weyl coefficient for the general case.

Idea of proof: Levitan's wave equation method

Key word: *microlocal analysis*.

Developed by B.M.Levitan, L.Hörmander (Fields Medal 1962), J.J.Duistermaat, V.W.Guillemin and others.

Introduce time t and study the "hyperbolic" equation

$$Au = \left(i\frac{\partial}{\partial t}\right)^{2m} u.$$

Construct the operator

$$U(t) := e^{-itA^{1/(2m)}}$$

This operator is called the *propagator*. It is a *Fourier integral operator*.

Having constructed the propagator, recover information about the spectrum using *Fourier Tauberian theorems*. These allow us to perform the inverse Fourier transform from variable t (time) to variable λ (spectral parameter) using incomplete information, with control of error terms.

Similar to Tauberian theorems used in analytic number theory.

Example: vibrations of a plate

$$\Delta^2 u = \lambda u \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \qquad u|_{\partial\Omega} = \partial u/\partial n|_{\partial\Omega} = 0.$$

My formula (1987):

$$N(\lambda) = \frac{S}{4\pi}\lambda^{1/2} + \frac{\beta L}{4\pi}\lambda^{1/4} + o(\lambda^{1/4}) \quad \text{as} \quad \lambda \to +\infty$$

where S is area of the plate, L is length of the boundary and

$$\beta = -1 - \frac{\Gamma(3/4)}{\sqrt{\pi}\,\Gamma(5/4)} \approx -1.763.$$

The first asymptotic term was derived by Courant (1922).

Inverting the formula and switching to frequencies $\lambda_N^{1/2}$, we get

$$\lambda_N^{1/2} = \frac{4\pi}{S}N - \frac{2\sqrt{\pi}\,\beta L}{S^{3/2}}\sqrt{N} + o(\sqrt{N}) \quad \text{as} \quad N \to +\infty.$$

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Current & Upcoming Events

August 08-10, 2018, Vladivostok
International Conference "**Fluxes and structures in fluids**".
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Past Events

November 21-24, 2017
11th International Conference "Aerophysics and Physical Mechanics of Classical and Quantum Systems" – APhM-2017.

November 7-9, 2017
Seven International Scientific School for Young Scientists "**Waves and vortices in complex media**".
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November 1-3, 2017
Third International Scientific School for Young Scientists "**Physical and mathematical modeling of processes in geomedia**".
Announcement Abstract template Registration form Program

October 2-7, 2017, Tsaghkadzor, Armenia
V International Conference on **Topical Problems of Continuum Mechanics** with a Special Session in Honor of Alexander Manzhirov's 60th Birthday
Site & Information: www.mechins.sci.am/tpcm2017/

Information

50th anniversary of the Institute for Problems in Mechanics of the Russian Academy of Sciences **1965–2015**

This is the new website of the Institute for Problems in Mechanics of the Russian Academy of Sciences. It was launched on 9th November 2015. The old website is available at http://www1.ipmnet.ru.

EqWorld – World of mathematical equations is internet-portal devoted to equations and methods of their solution.

MechMath. Mechanics and Applied Mathematics presents extensive information on

Krylov State Research Centre

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News

RSS

- 22 November 2017 Krylov Centre at International Scientific and Industrial Composite Forum
- 22 November 2017 Krylov Centre hosted the 1st All-Russian Scientific and Technical Conference of Young Scientists and Specialists «Scientific and technological development of shipbuilding»
- 30 October 2017 Krylov Centre and IAA PortNews held the conference «LNG Fleet and LNG Bunkering in Russia»

ВикипедиЯ

Климов, Дмитрий Михайлович

Материал из Википедии — свободной энциклопедии

Дми́трий Миха́йлович Кли́мов (род. 13 июля 1933, Лихославль) — советский и российский учёный-механик, специалист в области теоретической и прикладной механики, механики деформируемого твёрдого тела; автор трудов по механике гироскопических систем. Действительный член РАН (1992), доктор физико-математических наук (1965), профессор (1974). Лауреат Государственной премии СССР (1976) и Государственной премии Российской Федерации (1994).

Дмитрий Михайлович Климов

Дата рождения:	13 июля 1933 (84 года)
Место рождения:	Лихославль, Московская область, РСФСР, СССР
Страна:	СССР Россия
Научная сфера:	механика
Место работы:	Московский государственный университет, ИПМех РАН

Содержание

Биография

Окончил механико-математический факультет МГУ (1955), дипломную работу выполнил под руководством Н. Г. Четаева.

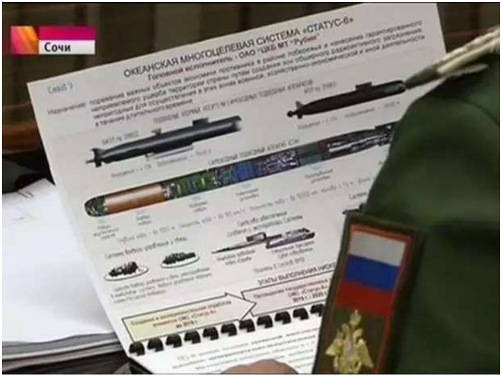
Работал в НИИ парашютных систем, Институте прикладной механики.

С 1967 года работает в Институте проблем механики АН СССР (ныне — РАН), с 1990 года по 2004 год — директор института.

Russian state-controlled TV 'accidentally' broadcasts secret plans for nuclear torpedo system

Vladimir Putin had announced the development of a system to defeat Nato defences

Lizzie Dearden @lizziedearden Thursday 12 November 2015 09:18 GMT 11 comments



Nuclear-Powered Ballistic Missile Submarines

Since 1956, Rubin's research and engineering personnel have had the leading role in concept, research and technical design in the program which produced about 100 strategic submarine cruisers.

Missile-carrying submarines were consistently modified mainly due to new missile systems, which promoted steady improvement of submarines' tactical and technical characteristics such as missile range, weaponry, firing accuracy, missile defence penetration capabilities, etc.

The first generation of nuclear-powered ballistic missile submarines – Project 658 (*Hotel*) – was constructed to the design of Rubin in 1960-1964.

Subsequently, nuclear-powered ballistic missile submarines known as Project 667A (*Yankee*), Project 667B (*Delta I*), Project 667BD (*Delta II*), Project 667BDR (*Delta III*), Project 667BDRM (*Delta IV*) were built to Rubin's designs.

Based on the previous experience, the team of Rubin jointly with other companies solved the challenging problem of producing the Typhoon seagoing strategic system on the basis of the third generation heavy nuclear underwater cruiser - Project 941 (*Typhoon*) - armed with solid-propellant multiple warhead ballistic missiles.

A great number of large-size missiles on board as well as the submarine length and draft limited by operating requirements called for a new design with a large reserve buoyancy. The new architecture made it possible to achieve unrivalled parameters for reliability, damage control, manoeuvrability and habitability at the same time. Remote automated control of the boat and her equipment was introduced, while living and service spaces were expanded to improve service and living conditions.

For the boat production, Rubin and PO «Sevmash» (Severodvinsk) introduced an innovative aggregate-and-modular method of construction. Subsequently it was widely adopted in shipbuilding.

For over 50 years of construction of nuclear-powered missile submarines a remarkable progress has been achieved in their technical level and efficiency.

Currently, a series of strategic missile submarine cruisers of the fourth generation - Project 955 designed by Rubin - is under construction at PO "Sevmash". The submarines will make the basis of the marine component of Russian strategic nuclear forces in the 21st century.

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ABOUT THE COMPANY

Joint-Stock Company “Scientific and Production Association “Orion” (JSC “SPA “Orion”) was founded in 1946 upon the initiative of S. I. Vavilov, Academician, President of the Academy of Sciences of the USSR. Since 1994, it is the State Scientific Center of the Russian Federation, the only one in the field of solid-state photoelectronics. In the period of its activity, JSC “SPA “Orion” has designed and manufactured thousands of new item types: optoelectronic converters, night vision devices, photoreceivers and photodetectors, thermal imaging devices, IR lasers, electron-beam and ion-plasma equipment, electronic microscopes, special calculators and other devices and units.

Central Institute of Aviation Motors named after P.I. Baranov is a world-class research center

Being the Russian Federation’s State Research Center, federal state unitary enterprise Central Institute of Aviation Motors named after P.I. Baranov is a world-class scientific center, an organization conducting the whole scope of research activity necessary for the development of the aircraft engines and gas turbine units based on them.

Operation Mole Cricket 19

Operation Mole Cricket 19 (Hebrew: מבצע ערצב 19, *Mivtza 'Artzav Tsha-Esreh*) was a suppression of enemy air defenses (SEAD) campaign launched by the Israeli Air Force (IAF) against Syrian targets on June 9, 1982, at the outset of the 1982 Lebanon War. The operation was the first time in history that a Western-equipped air force successfully destroyed a Soviet-built surface-to-air missile (SAM) network.[3] It also became one of the biggest air battles since World War II,[6] and the biggest since the Korean War.[7] The result was a decisive Israeli victory, leading to the colloquial name the "Bekaa Valley Turkey Shoot".

The IAF began working on a SAM suppression operation since the end of the Yom Kippur War. Rising tensions between Israel and Syria over Lebanon escalated in the early 1980s and culminated in Syria deploying the SAM batteries in the Beqaa Valley. On June 6, 1982, Israel invaded Lebanon, and on the third day of the war, with clashes going on between the Israel Defense Forces (IDF) and the Syrian Army, Israel decided to launch the operation.

The battle lasted about two hours, and involved innovative tactics and technology. By the end of the day, the IAF had destroyed 29 of 30 SAM batteries deployed in the Beqaa Valley and shot down between 82–86 enemy aircraft with minimal losses. The battle led the United States to impose a ceasefire on Israel and Syria.

Operation Mole Cricket 19

Part of the 1982 Lebanon War

Part of a Syrian SA-6 site built near the Beirut-Damascus highway, and overlooking the Beqaa Valley, in early 1982.

Date	June 9, 1982
Location	Beqaa Valley, Lebanon
Result	Decisive Israeli victory

Belligerents

Israel	Syria

Commanders and leaders

David Ivry Ariel Sharon	Mustafa Tlass Hafez Al-Assad

Strength

≈90 fighter aircraft (mostly F-15s and F-16s)[1] 1 remotely piloted vehicles squadron[2] UAVs	≈100 fighter aircraft (mostly ground attack MiG-21 and MiG-23, some MiG-23M air-to-air fighters)[1] 30 SAM batteries[3]

Casualties and losses

Contents

Юдин Анатолий Семенович

Д 212.208.06 - Член диссертационного совета

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Дополнительная информация:

Юдин Анатолий Семенович 1946 г. рождения. Окончил мехмат РГУ в 1968г. В 1974г. защитил кандидатскую диссертацию «Исследование устойчивости подкрепленных оболочек вращения при осесимметричной деформации и больших перемещениях», в 1991г. — докторскую по теме: "Развитие математических моделей и методов расчета вибраций и звукоизлучения конструктивно-сложных оболочек".

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My current research programme

Study of first order systems on manifolds without boundary.

Why do this?

- Second Weyl coefficient appears even without boundary. It comes from the bulk (interior) of the manifold.
- The spectrum is not semi-bounded. Spectral asymmetry.
- I developed an obsession in my old age: I want to understand elementary particles.

My main result for first order systems on manifolds

Explicit formula for second Weyl coefficient.

O.Chervova, R.J.Downes and D.Vassiliev, *The spectral function of a first order elliptic system*, Journal of Spectral Theory **3** (2013), 317–360.

Warning: doing microlocal analysis for systems is not easy

- V.Ivrii, 1980, Soviet Math. Doklady.
- V.Ivrii, 1982, Funct. Anal. Appl.
- G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.
- V.Ivrii, 1984, Springer Lecture Notes.
- Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.
- V.Ivrii, book, 1998, Springer.
- W.J.Nicoll, PhD thesis, 1998, University of Sussex.
- I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.

The circle group $\mathrm{U}(1)$

$$\mathrm{U}(1) = \{z \in \mathbb{C} : \ |z| = 1\}.$$

Here the group operation is multiplication.

Why the circle group $\mathrm{U}(1)$ is relevant

Look at the matrix differential operator A, keep only leading (first order) derivatives and replace each $\partial/\partial x^\alpha$ by $i\xi_\alpha$, $\alpha = 1, \dots, n$, to get a matrix-function $A_1(x, \xi)$ on the cotangent bundle.

The matrix-function $A_1(x, \xi)$ is called *principal symbol*.

Let $v(x, \xi)$ be an eigenvector of the principal symbol.

Problem: $v(x, \xi)$ is not defined uniquely. It is is defined modulo a gauge transformation $v \mapsto e^{i\phi} v$ where $\phi : T^*M \setminus \{0\} \to \mathbb{R}$ is an arbitrary smooth function. This gives rise to a $\mathrm{U}(1)$ connection which, in turn, generates curvature.

Physical meaning of the $\mathrm{U}(1)$ connection

In theoretical physics a $\mathrm{U}(1)$ connection is usually associated with electromagnetism. The corresponding curvature tensor is the electromagnetic (Faraday) tensor.

I have shown that inside **any** system of partial differential equations with variable coefficients there is an intrinsic electromagnetic field which lives on the cotangent bundle. Abstract mathematical fact.

One has to take account of this intrinsic electromagnetic field in order to get correct results.

Basic ideas driving my current research programme

- God is more of an analyst than a geometer.
- Dimension four is special.

Recent results

Suppose I am looking at a system of two linear first order PDEs for two unknown complex-valued scalar fields over a 4-manifold.

Suppose I know that this system admits a variational formulation.

Then Lorentzian geometry is automatically encoded within this system of PDEs. There is no need to introduce geometric constructs a priori. They are already there.

Z. Avetisyan, Y.-L. Fang, N. Saveliev and D. Vassiliev, *Analytic definition of spin structure*, JMP **58** (2017) 082301.

Why does Lorentzian metric appear out of thin air?

Observation: 2×2 Hermitian matrices form a real vector space of dimension four. Our manifold also has dimension four.

Take four linearly independent 2×2 Hermitian matrices σ^α, $\alpha = 1, 2, 3, 4$, multiply them by ξ_α, $\alpha = 1, 2, 3, 4$, and add up. This gives us a principal symbol

$$A_1(\xi) := \sigma^\alpha \xi_\alpha \,.$$

Determinant of principal symbol is a quadratic form in momentum

$$\det A_1(\xi) = -g^{\alpha\beta}\, \xi_\alpha \xi_\beta \,.$$

Lemma The real symmetric matrix $g^{\alpha\beta}$ has Lorentzian signature, i.e. it has three positive eigenvalues and one negative eigenvalue.