Classification of first order sesquilinear forms

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Main results in

Z. Avetisyan, Y.-L. Fang, N. Saveliev and D. Vassiliev, *Analytic definition of spin structure*, JMP **58** (2017) 082301.

Another paper in preparation.

Linear algebra in a finite-dimensional complex vector space

Concept of self-adjoint linear operator L. Requires inner product.

Concept of Hermitian sesquilinear form $S(u, v)$. Does not require inner product.

In the presence of an inner product there is a one-to-one correspondence between self-adjoint linear operators and sesquilinear forms: $S(u, v) = \langle u, Lv \rangle$.

Given an Hermitian sesquilinear form $S(u, v)$, can define a real-valued quadratic form $S(v, v)$. A physicist would call $S(v, v)$ an *action*. Variation gives a system of equations for v.

Object of study

Let M be a connected m-dimensional manifold without boundary, local coordinates $x = (x^1, \ldots, x^m)$.

Will work with n-columns $u : M \to \mathbb{C}^n$ of scalar fields.

First order sesquilinear form:

$$S(u,v) := \int_M \left[u^* \mathbf{A}^\alpha \left(\frac{\partial v}{\partial x^\alpha} \right) + \left(\frac{\partial u}{\partial x^\alpha} \right)^* \mathbf{B}^\alpha v + u^* \mathbf{C} v \right] dx \,,$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrix-valued vector densities, $\mathbf{C}$ is a matrix-valued density and $dx = dx^1 \ldots dx^m$.

I reserve bold font for density-valued quantities.

The symbol of a first order sesquilinear form

Canonical representation of a first order sesquilinear form:

$$S(u,v)=\int_M\left[-\frac{i}{2}u^*\mathbf{E}^\alpha\left(\frac{\partial v}{\partial x^\alpha}\right)+\frac{i}{2}\left(\frac{\partial u}{\partial x^\alpha}\right)^*\mathbf{E}^\alpha v+u^*\mathbf{F}v\right]dx\,.$$

Density-valued principal symbol $\mathbf{S}_{\mathrm{prin}}(x,p):=\mathbf{E}^\alpha(x)\,p_\alpha\,.$

Density-valued subprincipal symbol $\mathbf{S}_{\mathrm{sub}}(x):=\mathbf{F}(x)\,.$

Density-valued full symbol $\mathbf{S}_{\mathrm{full}}(x,p):=\mathbf{S}_{\mathrm{prin}}(x,p)+\mathbf{S}_{\mathrm{sub}}(x)\,.$

Full symbol uniquely determines the sesquilinear form.

Sesquilinear form is Hermitian iff its full symbol is Hermitian.

Non-degeneracy condition

Definition 1 We say that our Hermitian first order sesquilinear form S is *non-degenerate* if

$$\mathbf{S}_{\mathrm{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}.$$

Gauge transformations of sesquilinear forms

Consider a smooth matrix-function

$$R : M \to GL(n, \mathbb{C}).$$

Given a sesquilinear form S can define another sesquilinear form

$$\tilde{S}(u, v) := S(Ru, Rv).$$

The corresponding full symbol is

$$\tilde{\mathbf{S}}_{\mathrm{full}} = R^* \, \mathbf{S}_{\mathrm{full}} \, R + \frac{i}{2} \left[R^*_{x^\alpha} (\mathbf{S}_{\mathrm{full}})_{p_\alpha} R - R^* (\mathbf{S}_{\mathrm{full}})_{p_\alpha} R_{x^\alpha} \right].$$

Want to solve 'inverse problem'. We are given two full symbols, $\mathbf{S}_{\mathrm{full}}$ and $\tilde{\mathbf{S}}_{\mathrm{full}}$. Do they describe the same sesquilinear form?

Equivalence classes of symbols

Definition 2 We say that two full symbols $\mathbf{S}_{\mathrm{full}}$ and $\tilde{\mathbf{S}}_{\mathrm{full}}$ are *GL-equivalent* if there exists a smooth matrix-function

$$R : M \to GL(n, \mathbb{C}). \tag{1}$$

such that

$$\tilde{\mathbf{S}}_{\mathrm{full}} = R^* \, \mathbf{S}_{\mathrm{full}} \, R + \frac{i}{2} \left[R^*_{x^\alpha} (\mathbf{S}_{\mathrm{full}})_{p_\alpha} R - R^* (\mathbf{S}_{\mathrm{full}})_{p_\alpha} R_{x^\alpha} \right]. \tag{2}$$

Definition 3 We say that two full symbols $\mathbf{S}_{\mathrm{full}}$ and $\tilde{\mathbf{S}}_{\mathrm{full}}$ are *SL-equivalent* if there exists a smooth matrix-function

$$R : M \to SL(n, \mathbb{C}). \tag{3}$$

such that (2) is satisfied.

Task at hand

Give explicit necessary and sufficient conditions for a pair of full symbols to be GL-equivalent or SL-equivalent.

I want to describe equivalence classes of sesquilinear forms.

Will achieve this goal for special values of m and n. Here m is the dimension of the manifold and n is the number of scalar fields.

The analysis that follows is dimension sensitive.

For definiteness will deal with SL-equivalence.

Special case $m = n^2$

Lemma 1 A manifold M admits a non-degenerate Hermitian first order sesquilinear form iff it is parallelizable.

Proof is based on the observation that $n \times n$ Hermitian matrices form a real vector space of dimension n^2. But $m = n^2$ is also the dimension of our manifold.

Case $m = 4$, $n = 2$: appearance of Lorentzian geometry

The determinant of the denisty-valued principal symbol is a quadratic form in momentum

$$\det \mathbf{L}_{\mathrm{prin}}(x, p) = -\mathbf{g}^{\alpha\beta}(x)\, p_\alpha p_\beta \,,$$

where $\mathbf{g}^{\alpha\beta}(x)$ is a real symmetric 4×4 matrix-function with values in 2-densities.

Lemma 2 The matrix-function $\mathbf{g}^{\alpha\beta}(x)$ has Lorentzian signature, i.e. it has three positive eigenvalues and one negative eigenvalue.

Definition of Lorentzian density: $\rho(x) := |\det \mathbf{g}^{\mu\nu}(x)|^{1/6}$. This density is invariant under gauge transformations.

Rewriting sesquilinear form in terms of half-densities

Turn scalar fields into half-densities: $v \mapsto \sqrt{\rho} v$. Our sequilinear form now reads

$$S(u,v) = \int_M \left[-\frac{i}{2} u^* E^\alpha \left(\frac{\partial v}{\partial x^\alpha} \right) + \frac{i}{2} \left(\frac{\partial u}{\partial x^\alpha} \right)^* E^\alpha v + u^* F v \right] dx \, .$$

Elements of the matrix E are vector fields and elements of the matrix F are scalar fields.

Principal symbol $S_{\text{prin}}(x,p) := E^\alpha(x)\, p_\alpha$. Invariantly defined 2×2 Hermitian matrix-function on T^*M.

Subprincipal symbol $S_{\text{sub}}(x) := F(x)$. Invariantly defined Hermitian 2×2 matrix-function on M.

Full symbol $S_{\text{full}}(x,p) := S_{\text{prin}}(x,p) + S_{\text{sub}}(x)$.

My definition of the metric tensor

$$\det S_{\mathrm{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta\,. \tag{4}$$

Metric is Lorentzian and is invariant under gauge transformations.

Time-orientability

The Lorentzian manifold (M, g) is said to be *time-orientable* if it admits a timelike vector field.

Lemma 3 A parallelizable Lorentzian manifold (M, g) admits a non-degenerate Hermitian first order sesquilinear form satisfying condition (4) iff it is time-orientable.

Proof in one direction is easy: just take trace of principal symbol.

Other way round not 100% obvious: proof relies on the fact that $\mathbb{S}^3$ is parallelizable.

Topological charge

$$c_{\mathrm{top}} := -\frac{i}{2}\sqrt{|\det g_{\alpha\beta}|}\ \mathrm{tr}\big((S_{\mathrm{prin}})_{p_1}(S_{\mathrm{prin}})_{p_2}(S_{\mathrm{prin}})_{p_3}(S_{\mathrm{prin}})_{p_4}\big),$$

where the subscripts p_1, p_2, p_3 and p_4 indicate partial derivatives with respect to the components of momentum.

Can take only two values, $+1$ or -1, and describes the orientation of the principal symbol relative to our chosen orientation of local coordinates $x = (x^1, x^2, x^3, x^4)$.

It is invariant under gauge transformations.

Temporal charge

$$\mathbf{c}_{\mathrm{tem}} := \operatorname{sgn} \operatorname{tr} S_{\mathrm{prin}}(x, q(x)) \,,$$

where q is a prescribed timelike covector field used as a reference.

Can take only two values, $+1$ or -1, and describes the orientation of the principal symbol relative to our chosen time orientation.

It is invariant under gauge transformations.

Spin structure

Definition 4 Consider two principal symbols, S_{prin} and $\tilde{S}_{\mathrm{prin}}$, which carry the same metric, same topological charge and same temporal charge. We say that these two principal symbols are *spin-equivalent* if we have

$$\tilde{S}_{\mathrm{prin}} = R^* S_{\mathrm{prin}} R$$

for some smooth matrix-function $R : M \to SL(2, \mathbb{C})$. An equivalence class of principal symbols is called *spin structure*.

Lemma 4 For parallelizable time-orientable Lorentzian 4-manifolds the two definitions of spin structure, our analytic definition and the traditional one, are equivalent.

Proof due to Nikolai Saveliev.

How restrictive is the parallelizability assumption?

Lemma 5 A non-compact time-orientable Lorentzian 4-manifold is parallelizable if and only if it is spin.

Proof also due to Nikolai Saveliev.

Implication: my analytic definition of spin structure may not work for Lorentzian 4-manifolds that are compact or not time-orientable.

Dealing with the subprincipal symbol

Subprincipal symbol transforms as

$$S_{\mathrm{sub}} \mapsto R^* S_{\mathrm{sub}} R + \frac{i}{2} \left(R^*_{x^\alpha} (S_{\mathrm{prin}})_{p_\alpha} R - R^* (S_{\mathrm{prin}})_{p_\alpha} R_{x^\alpha} \right).$$

Problem: subprincipal symbol does not transform covariantly.

Solution: define *covariant* subprincipal symbol $S_{\mathrm{csub}}(x)$ as

$$S_{\mathrm{csub}} := S_{\mathrm{sub}} + \frac{i}{16} g_{\alpha\beta} \{S_{\mathrm{prin}}, \operatorname{adj} S_{\mathrm{prin}}, S_{\mathrm{prin}}\}_{p_\alpha p_\beta},$$

where $\{U, V, W\} := U_{x^\alpha} V W_{p_\alpha} - U_{p_\alpha} V W_{x^\alpha}$ is the generalised Poisson bracket on matrix-functions and adj is the operator of matrix adjugation $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \operatorname{adj} U$.

Electromagnetic covector potential appears out of thin air

The covariant subprincipal symbol can be uniquely represented as

$$S_{\mathrm{csub}}(x) = S_{\mathrm{prin}}(x, A(x)), \tag{5}$$

where A is a real-valued covector field which is invariant under gauge transformations.

Explanation: the matrices $(S_{\mathrm{prin}})_{p_\alpha}$, $\alpha = 1, 2, 3, 4$, are Pauli matrices and these form a basis in the real vector space of 2×2 Hermitian matrices. Formula (5) is simply an expansion of the matrix S_{csub} with respect to the basis of Pauli matrices.

Main result

Theorem 1 A pair of full symbols is SL-equivalent iff

- their metrics are the same,
- their electromagnetic covector potentials are the same,
- their topological charges are the same,
- their temporal charges are the same and
- they have the same spin structure.

Bottom line, in plain English

Suppose that I am looking at a system of two linear first order PDEs for two unknown complex-valued scalar fields over a 4-manifold.

Suppose that I know that this system of PDEs admits a variational formulation.

Then Lorentzian geometry is automatically encoded within this system of PDEs.

There is no need to introduce geometric constructs a priori. They are already there.

3-dimensional Riemannian geometry

1. More restrictive choice of sesquilinear forms: $\operatorname{tr} S_{\mathrm{prin}}(x,p) = 0$.

2. My non-degeneracy condition is now equivalent to the more familiar ellipticity condition $\det S_{\mathrm{prin}}(x,p) \neq 0$.

3. A 3-manifold admits a 2×2 first order sesquilinear form with trace-free principal symbol iff it is parallelizable.

4. A 3-manifold is parallelizable iff it is orientable.

5. My metric is automatically Riemannian: $\det S_{\mathrm{prin}}(x,p) < 0$.

6. More restrictive choice of gauge transformations:

$$R : M \to SU(2).$$

Appearance of an operator

In the 3-dimensional setting we have a natural inner product on 2-columns of complex-valued half-densities:

$$\langle u, v \rangle := \int_M u^* v \, dx \, .$$

Our gauge transformations preserve this inner product.

An Hermitian sesquilinear form S can now be identified with a self-adjoint linear operator L via the formula $S(u,v) = \langle u, Lv \rangle$.

Definition 5 A *massless Dirac operator* is an elliptic self-adjoint 2×2 first order linear differential operator with trace-free principal symbol and zero covariant subprincipal symbol.

Examples from 3-dimensional Riemannian geometry

1. $\mathbb{S}^3$ has a unique spin structure.

2. $\mathbb{T}^3$ has eight distinct spin structures.

Two different spin structures on $\mathbb{T}^3$

Using cyclic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π:

$$L_{\mathrm{prin}}(x, p) = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix},$$

$$\begin{aligned} L_{\mathrm{prin}}(x, p) &= \begin{pmatrix} p_3 & e^{ix^3}(p_1 - ip_2) \\ e^{-ix^3}(p_1 + ip_2) & -p_3 \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{i}{2}x^3} & 0 \\ 0 & e^{-\frac{i}{2}x^3} \end{pmatrix} \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \begin{pmatrix} e^{-\frac{i}{2}x^3} & 0 \\ 0 & e^{\frac{i}{2}x^3} \end{pmatrix}. \end{aligned}$$

Special unitary matrix-function in latter formula is discontinuous.

Two different massless Dirac operators on $\mathbb{T}^3$

$$L = -i \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix},$$

$$L = -i \begin{pmatrix} \partial_3 & e^{ix^3}(\partial_1 - i\partial_2) \\ e^{-ix^3}(\partial_1 + i\partial_2) & -\partial_3 \end{pmatrix} - \frac{1}{2} I \,.$$

Their spectra are different.