Lorentzian elasticity
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Abstract formulation of elasticity theory

Consider a manifold M equipped with non-degenerate metric g.

Unknown quantity in elasticity is diffeomorphism $\varphi : M \to M$.

Perturbed metric

$$h_{\alpha\beta}(x) := g_{\mu\nu}(\varphi(x)) \frac{\partial \varphi^\mu}{\partial x^\alpha} \frac{\partial \varphi^\nu}{\partial x^\beta} \,,$$

where

$$y = \varphi(x)$$

is the representation of the diffeomorphism in local coordinates.

A pair of metrics, g and h, allows us to write down an action (variational functional).

Strain tensor

Linear algebra: a pair of non-degenerate symmetric bilinear forms $g, h : V \times V \to \mathbb{R}$ in a real finite-dimensional vector space V defines an invertible linear operator $L : V \to V$ via the formula

$$h(u, v) = g(Lu, v), \qquad \forall u, v \in V.$$

Convenient to subtract the identity operator,

$$S := L - \mathrm{Id}$$

Definition of strain tensor:

$$S^{\alpha}{}_{\beta}(x) := [g^{\alpha\gamma}(x)]\,[h_{\gamma\beta}(x)] - \delta^{\alpha}{}_{\beta}\,.$$

Describes, pointwise, linear map in the fibres of the tangent bundle

$$v^{\alpha} \mapsto S^{\alpha}{}_{\beta}\, v^{\beta}.$$

Scalar invariants of the strain tensor

Obvious choice: $\mathrm{tr}(S^k)$, $k = 1, \ldots, d$, where d is dimension of M.

More convenient choice:

$$\begin{aligned}
e_1(\varphi) &:= \mathrm{tr}\, S = \lambda_1 + \ldots + \lambda_d\,, \\
e_2(\varphi) &:= \frac{1}{2}\left[(\mathrm{tr}\, S)^2 - \mathrm{tr}(S^2)\right] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{d-1}\lambda_d\,, \\
&\ldots \\
e_d(\varphi) &:= \det S = \lambda_1\, \ldots\, \lambda_d\,.
\end{aligned}$$

These are elementary symmetric polynomials. The lambdas are the eigenvalues of the strain tensor.

Elastic action

$$\mathcal{J}(\varphi) := \int_M \mathcal{L}\big(e_1(\varphi), \ldots, e_d(\varphi)\big) \sqrt{|\det g_{\mu\nu}(x)|}\, dx\,,$$

where $\mathcal{L}$ is some smooth real-valued function of d real variables and $dx := dx^1 \ldots dx^d$.

This action describes an isotropic homogeneous elastic continuum.

Examples of meaningful Lagrangians

Example 1 Lagrangian linear in strain

$$\mathcal{L}(e_1, \ldots, e_d) = e_1 .$$

This is the Lagrangian of a harmonic map. No free parameters.

Example 2 Lagrangian quadratic in strain

$$\mathcal{L}(e_1, \ldots, e_d) = \alpha (e_1)^2 + \beta e_2 ,$$

where $\beta \neq 0$. This is the Lagrangian of an elastic continuum that is physically linear. One free parameter: $\dfrac{\alpha}{\beta}$.

Poisson's ratio: $\nu = \dfrac{2\alpha + \beta}{4\alpha + \beta}$.

Equations of elasticity

$$E(\varphi) = 0,$$

where E is a nonlinear second order partial differential operator mapping a diffeomorphism to a covector field.

Lorentzian elasticity

$d = 4$, metric has signature $+ + + -$.

Remark 1 Lorentzian elasticity has never been studied. Not to be confused with 'relativistic elasticity'.

Remark 2 In the Lorentzian setting strain may be nilpotent with nilpotency index ≤ 3 (Gohberg, Lancaster and Rodman 2005). This means that a Lorentzian spacetime can be deformed in a most unusual way: strain is nonzero but all the scalar invariants are zero.

Our mathematical model

Vary elastic action subject to the volume preservation constraint

$$\det g_{\alpha\beta}(x) = \det h_{\mu\nu}(x).$$

Work in the subgroup of volume-preserving diffeomorphisms.

Equivalent representation of the volume preservation constraint:

$$e_1(\varphi) + e_2(\varphi) + e_3(\varphi) + e_4(\varphi) = 0\,.$$

Our field equations

Introducing a Lagrange multiplier $p : M \to \mathbb{R}$, we get

$$E(\varphi) - \mathrm{d}p = 0,$$

where $\mathrm{d}p$ is the gradient of p

One can interpret p as pressure, like in Stokes equations.

Conditions on the Lagrangian $\mathcal{L}(e_1, e_2, e_3, e_4)$

Put $f(z) := \mathcal{L}(-z, z, 0, 0)$, $f : \mathbb{R} \to \mathbb{R}$.

Condition 1 $f'(0) \neq 0$.

Condition 2 $f'(c) = 0$ for some $c \in (0, 4)$.

Vector field of displacements $A : M \to TM$

Connect a point $P \in M$ with the point $\varphi(P) \in M$ by a geodesic $\gamma : [0, 1] \to M$, so that $\gamma(0) = P$ and $\gamma(1) = \varphi(P)$. Parameterize the geodesic in such a way that $\gamma(\tau)$ is a solution of the equation

$$\ddot{\gamma}^{\lambda} + \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \dot{\gamma}^{\mu} \dot{\gamma}^{\nu} = 0,$$

where the dot stands for differentiation in τ.

Then

$$A(P) := \dot{\gamma}(0).$$

Linearised field equations

Theorem 1 Our linearised field equations read

$$\begin{pmatrix} \delta \mathrm{d} - 2\,\mathrm{Ric} & \mathrm{d} \\ \delta & 0 \end{pmatrix} \begin{pmatrix} A^\flat \\ \tilde{p} \end{pmatrix} = 0.$$

Here $A^\flat : M \to T^*M$ is the covector version of our vector field of displacements $A : M \to TM$ and $\tilde{p}$ is a rescaled version of our original Lagrange multiplier $p : M \to \mathbb{R}$ (pressure).

Remark 3 For Ricci-flat manifolds we get Maxwell's equations

$$\delta \mathrm{d} A^\flat = J$$

in the Lorenz gauge

$$\delta A^\flat = 0$$

with exact current $J := -\mathrm{d}\tilde{p}$.

Solving nonlinear equations: group-theoretic trick

Definition 1 Let φ be a diffeomorphism. We say that φ is *homogeneous* if there exists a subgroup H of the isometry group acting transitively on M and satisfying

$$H \circ \varphi = \varphi \circ H.$$

If we have the stronger property

$$\xi \circ \varphi = \varphi \circ \xi, \qquad \forall \xi \in H,$$

we say that φ is *equivariant*.

Theorem 2 Let φ be a homogeneous diffeomorphism. Then all the scalar invariants are constant. Furthermore, if the equations of elasticity are satisfied at a single point then they are satisfied on the whole manifold.

Minkowski space

Work in Minkowski space $\mathbb{M}$, $g_{\alpha\beta} = \mathrm{diag}(1, 1, 1, -1)$.

10-dimensional isometry group of $\mathbb{M}$, the Poincaré group.

Restricted Poincaré group (identity component of Poincaré group).

Need to find nontrivial 4-dimensional subgroups of the restricted Poincaré group.

Massless screw groups, right-handed and left-handed

This is the set of isometries

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \mapsto \begin{pmatrix} \cos(q^3+q^4) & \mp\sin(q^3+q^4) & 0 & 0 \\ \pm\sin(q^3+q^4) & \cos(q^3+q^4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} + \begin{pmatrix} q^1 \\ q^2 \\ q^3 \\ q^4 \end{pmatrix}, \quad q \in \mathbb{R}^4.$$

Massive screw group

For a given value of parameter $m > 0$, this is the set of isometries

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \mapsto \begin{pmatrix} \cos(2mq^4) & -\sin(2mq^4) & 0 & 0 \\ \sin(2mq^4) & \cos(2mq^4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} + \begin{pmatrix} q^1 \\ q^2 \\ q^3 \\ q^4 \end{pmatrix}, \quad q \in \mathbb{R}^4.$$

Explicit massless solutions of our nonlinear field equations

Theorem 3 Let $a > 0$ be parameter. Then the diffeomorphism described by the vector field of displacements

$$A^\alpha(x) = a \begin{pmatrix} \cos(x^3 + x^4) \\ \pm\sin(x^3 + x^4) \\ 0 \\ 0 \end{pmatrix}$$

is volume preserving and satisfies nonlinear equations of elasticity.

Explicit massive solutions of our nonlinear field equations

Theorem 4 Let $m > 0$, $a > 0$ and $b \in \mathbb{R}$ be parameters satisfying

$$4m^2(a^2 + b^2) = c\,,$$

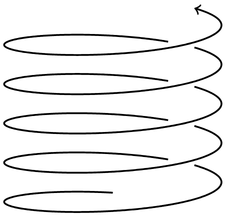
where c is the critical point from an earlier slide. Then the diffeomorphism described by the vector field of displacements

$$A^\alpha(x) = \begin{pmatrix} a\cos(2mx^4) \\ a\sin(2mx^4) \\ 2mbx^4 \\ 0 \end{pmatrix}$$

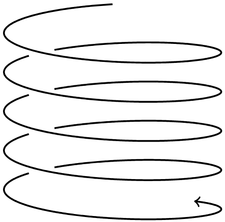
is volume preserving and satisfies nonlinear equations of elasticity.

Remark 4 We have here two free parameters, $m > 0$ and $b \in \mathbb{R}$.

- m is quantum mechanical mass.
- b may be interpreted as electric charge.



(i) $b > 0$



(ii) $b < 0$

Figure 1: Massive solution

The Dirac equation

Claim One can see the Dirac equation in our explicit solutions.

Need to separate stretches and rotations.

Deformation gradient

$$D^{\alpha}{}_{\beta} := \delta^{\alpha}{}_{\beta} + \frac{\partial A^{\alpha}}{\partial x^{\beta}} \,.$$

It contains more geometric information than the strain tensor.

Polar decomposition

$$D^{\alpha}{}_{\beta} = U^{\alpha}{}_{\gamma} \, V^{\gamma}{}_{\beta} \,,$$

where U is Lorentz–orthogonal and V is Lorentz–symmetric.

Define the rotation 2-form as $F := \ln U$.

For our explicit solutions the rotation 2-form F admits a natural complexification $\mathbb{F}$,

$$F = \operatorname{Re} \mathbb{F}.$$

Turns out, this complex-valued 2-form is degenerate,

$$\det \mathbb{F} = \det(*\mathbb{F}) = 0,$$

therefore it is equivalent to the square of a bispinor field.

Turns out, this bispinor field satisfies the Dirac equation.