Geometric wave propagator on Riemannian manifolds

Matteo Capoferri, Michael Levitin and Dmitri Vassiliev

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Playing field

Let (M, g) be a closed Riemannian manifold of dimension $d \geq 2$.
Local coordinates x^α, $\alpha = 1, \ldots, d$.

Will work with scalar functions $u : M \to \mathbb{C}$.

Inner product

$$(u, v) := \int_M \overline{u(x)}\, v(x)\, \rho(x)\, dx\,,$$

where $\rho(x) := \sqrt{\det g_{\mu\nu}(x)}$ and $dx = dx^1 \ldots dx^d$.

Laplace–Beltrami operator

$$\Delta := \rho(x)^{-1} \frac{\partial}{\partial x^\mu} \rho(x)\, g^{\mu\nu}(x) \frac{\partial}{\partial x^\nu} .$$

Eigenvalues and normalised eigenfunctions:

$$-\Delta v_k = \lambda_k^2 v_k ,$$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \to +\infty.$$

Cauchy problem for the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) f(t,x) = 0\,,$$

$$f(0,x) = f_0(x), \qquad \frac{\partial f}{\partial t}(0,x) = f_1(x).$$

Solution:

$$f = \cos\left(t\sqrt{-\Delta}\,\right) f_0 + \sin\left(t\sqrt{-\Delta}\,\right)(-\Delta)^{-1/2} f_1 + t\,(v_0\,, f_1)\,,$$

where

$$(-\Delta)^{-1/2} := \sum_{k=1}^{\infty} \frac{1}{\lambda_k}\,(v_k\,,\,\cdot\,)$$

is the pseudoinverse of the operator $\sqrt{-\Delta}$.

The propagator

$$U(t) := \cos\left(t\sqrt{-\Delta}\right) - i\sin\left(t\sqrt{-\Delta}\right) = e^{-it\sqrt{-\Delta}} = \int u(t,x,y)\,(\,\cdot\,)\,\rho(y)\,dy\,,$$

whose Schwartz kernel reads

$$u(t,x,y) := \sum_{k=0}^{\infty} e^{-it\lambda_k}\, v_k(x)\,\overline{v_k(y)}\,.$$

Propagator is the solution of the operator-valued Cauchy problem

$$\left(-i\,\frac{\partial}{\partial t} + \sqrt{-\Delta}\right) U(t) = 0\,,$$

$$U(0) = \mathrm{Id}\,.$$

Example: the propagator for Euclidean space

Consider $M = \mathbb{R}^d$ equipped with Cartesian coordinates and metric $g_{\alpha\beta}(x) = \delta_{\alpha\beta}$. The Schwartz kernel of the propagator reads

$$u(t, x, y) = \frac{1}{(2\pi)^d} \int e^{i\varphi(t,x;y,\eta)}\, d\eta\,,$$

where

$$\varphi : \mathbb{R} \times M \times T^*M, \qquad \varphi(t, x; y, \eta) = (x - y)^\alpha \eta_\alpha - t\|\eta\|$$

and $d\eta = d\eta_1 \dots d\eta_d$.

Goal

Write Schwartz kernel of the propagator explicitly, modulo an infinitely smooth function, for a general Riemannian manifold.

'Explicitly' means reduction to ODEs.

Basic idea: replace

$$\varphi(t,x;y,\eta)=(x-y)^\alpha\eta_\alpha-t\|\eta\|$$

by a different phase function. Phase function has to feel the geometry of the particular Riemannian manifold (M,g).

Invariant construction global in space and time

Want a single invariantly defined oscillatory integral.

Don't want to work in coordinate charts and patch them together.

Don't want to take compositions in time.

A. Laptev, Yu. Safarov and D. Vassiliev, On global representation of Lagrangian distributions and solutions of hyperbolic equations, *Comm. Pure Appl. Math.* **47** 11 (1994) 1411–1456.

Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*. Amer. Math. Soc., Providence (RI), 1997, 1998.

Our contribution: detailed analysis for Riemannian manifolds.

Metric gives additional structure.

Hamiltonian flow

$$h(x,\xi) := \sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\,\xi_\beta} = \|\xi\|\,,$$

$$\begin{cases} \dot{x}^* = h_\xi(x^*,\xi^*)\,, \\ \dot{\xi}^* = -h_x(x^*,\xi^*)\,, \end{cases}$$

$$(x^*,\xi^*)|_{t=0} = (y,\eta)\,.$$

Hamiltonian trajectories

$$(x^*(t;y,\eta),\xi^*(t;y,\eta))$$

play the role of a skeleton in our construction.

The Levi-Civita phase function

For x close to $x^*(t; y, \eta)$

$$\varphi(t, x; y, \eta) := \int_\gamma \zeta \, dz + \frac{i\,\epsilon}{2} \, h(y, \eta) \, \mathrm{dist}^2(x, x^*(t; y, \eta)) \, ,$$

where the path of integration γ is the (unique) shortest geodesic connecting $x^*(t; y, \eta)$ to x, ζ is the result of the parallel transport of $\xi^*(t; y, \eta)$ along γ and $\epsilon > 0$ is a parameter.

The phase function is chosen to be complex-valued so that

$$\det \varphi_{x^\alpha \eta_\beta} \big|_{x = x^*} \neq 0 \, .$$

Schwartz kernel of the propagator as an oscillatory integral

$$u(t,x,y) \overset{\mathrm{mod}\, C^\infty}{=} \frac{1}{(2\pi)^d} \int e^{i\varphi(t,x;y,\eta)}\, \mathfrak{a}(t;y,\eta)\, \chi(t,x;y,\eta)\, w(t,x;y,\eta)\, d\eta\,,$$

where

- $\mathfrak{a}$ is the global invariantly defined full symbol,
- χ is a cut-off and
- w is a weight defined as

$$w(t,x;y,\eta) := [\rho(x)]^{-1/2}\, [\rho(y)]^{-1/2} \left[\det{}^2\left(\varphi_{x^\alpha \eta_\beta}(t,x;y,\eta)\right)\right]^{1/4}.$$

Calculating the full symbol of the propagator

Expansion into components positively homogeneous in η, subscript indicates degree of homogeneity:

$$\mathfrak{a}(t; y, \eta) \sim \sum_{k=0}^{\infty} \mathfrak{a}_{-k}(t; y, \eta) \,.$$

What we have to date.

- Principal symbol $\mathfrak{a}_0(t; y, \eta) = 1$.
- Explicit formula for the subprincipal symbol $\mathfrak{a}_{-1}(t; y, \eta)$. Very long formula on next two slides.
- Algorithm for the calculation of $\mathfrak{a}_{-k}(t; y, \eta)$, $k = 2, 3, \ldots$.

Theorem 1 (Capoferri–Levitin–Vassiliev) The global invariantly defined subprincipal symbol of the propagator is

$$\mathfrak{a}_{-1}(t;y,\eta) = -\frac{i}{2\,h}\int_0^t \left[\mathfrak{S}_{-2}\, b_2 + \mathfrak{S}_{-1}\, b_1 + \mathfrak{S}_0\, b_0\right](\tau;y,\eta)\, d\tau\,,$$

where the functions b_k, $k = 0, 1, 2$, are defined as

$$b_0 = w^{-1}\left[w_{tt} - g^{\alpha\beta}(x)\,\nabla_\alpha\nabla_\beta\, w\right],$$

$$b_1 = i\,\left[\varphi_{tt} - g^{\alpha\beta}(x)\,\nabla_\alpha\nabla_\beta\,\varphi + 2\,(\log w)_t\,\varphi_t - 2\,g^{\alpha\beta}(x)\,[\nabla_\alpha(\log w)]\,\nabla_\beta\,\varphi\right],$$

$$b_2 = -\,(\varphi_t)^2 + g^{\alpha\beta}(x)\,(\nabla_\alpha\varphi)(\nabla_\beta\varphi)\,,$$

and the operators $\mathfrak{S}_{-k}$, $k = 0, 1, 2$, are defined as

$$\mathfrak{S}_0 := (\,\cdot\,)\big|_{x=x^*}\ ,$$

$$L_\alpha := \left[(\varphi_{x\eta})^{-1}\right]_\alpha{}^\beta\,\frac{\partial}{\partial x^\beta}\,,$$

$$\mathfrak{B}_{-1} := i\,w^{-1}\,\frac{\partial}{\partial \eta_\alpha}\,w\,L_\alpha - \frac{i}{2}\,\varphi_{\eta_\alpha\eta_\beta}\,L_\alpha\,L_\beta\,,$$

$$\mathfrak{S}_{-1} = \mathfrak{S}_0\,\mathfrak{B}_{-1}\,,$$

$$\mathfrak{S}_{-2} = \mathfrak{S}_0\,\mathfrak{B}_{-1}\left[i\,w^{-1}\frac{\partial}{\partial \eta_\beta}\,w\left(1+\sum_{1\le|\boldsymbol{\alpha}|\le 3}\frac{(-\varphi_\eta)^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!(|\boldsymbol{\alpha}|+1)}\,L_{\boldsymbol{\alpha}}\right)L_\beta\right].$$

Bold Greek letters denote multi-indices in $\mathbb{N}_0^d$, $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d)$, $|\boldsymbol{\alpha}|=\sum_{j=1}^d \alpha_j$ and $(-\varphi_\eta)^{\boldsymbol{\alpha}} := (-1)^{|\boldsymbol{\alpha}|}\,(\varphi_{\eta_1})^{\alpha_1}\ldots(\varphi_{\eta_d})^{\alpha_d}$.

Subprincipal symbol of the propagator on a 2-sphere

For general $\epsilon > 0$

$$\mathfrak{a}_{-1}(t;y,\eta) = \frac{i\,t}{8\,\|\eta\|} + \frac{i\sin(2t) - 4\epsilon\sin^2(t) + 3\,i\,\epsilon^2\sin(2t) + 6\epsilon^3\sin^2(t)}{48\,\|\eta\|\,(\cos(t) - i\epsilon\sin(t))^2}\,.$$

If we take $\epsilon = 1$ formula simplifies and reads

$$\mathfrak{a}_{-1}(t;y,\eta) = \frac{i\,t}{8\,\|\eta\|} + \frac{2e^{2\,i\,t} + 3e^{4\,i\,t} - 5}{96\,\|\eta\|}\,.$$

For $\epsilon = 0$ formula becomes

$$\mathfrak{a}_{-1}(t;y,\eta) = \frac{i}{24\,\|\eta\|}\,(3t + \tan(t))\,.$$

Subprincipal symbol for the hyperbolic plane

Setting $\epsilon = 0$ we get

$$\mathfrak{a}_{-1}(t; y, \eta) = -\frac{i}{24\,\|\eta\|}\left(3t + \tanh(t)\right).$$

Invariant definition of the full symbol of a ΨDO

Full symbol of the identity operator:

$$\mathfrak{s}(y,\eta) = 1 + \frac{(d-1)\,(d-2)\,\epsilon^2}{8\,\|\eta\|^2} + \dots\,.$$

For $\epsilon = 0$ formula becomes

$$\mathfrak{s}(y,\eta) = 1\,.$$

Small time expansion of subprincipal symbol of propagator

Theorem 2 (Capoferri–Levitin–Vassiliev) Let $\epsilon = 0$. Then

$$\mathfrak{a}_{-1}(t; y, \eta) = \frac{i}{12\,\|\eta\|}\,\mathcal{R}(y)\,t + O(t^2) \quad \text{as} \quad t \to 0\,,$$

where $\mathcal{R}$ is scalar curvature.

This theorem allows us to recover the third Weyl coefficient in the asymptotic expansion of the local counting function:

$$N(y, \lambda) := \sum_{\lambda_k < \lambda} \|v_k(y)\|^2\,,$$

$$(N' * \mu)(y, \lambda) = a_{d-1}(y)\,\lambda^{d-1} + a_{d-2}(y)\,\lambda^{d-2} + a_{d-3}(y)\,\lambda^{d-3} + \ldots$$

as $\lambda \to +\infty$. Here $\mu(\lambda)$ is a mollifier.

Circumventing topological obstructions

Fix a point $y \in M$ and consider the one-parameter family of d-dimensional smooth submanifolds of the cotangent bundle:

$$\mathcal{T}_y(t) := \{(x^*(t; y, \eta), \xi^*(t; y, \eta)) \in T^*M \,|\, \eta \in T_y^*M \setminus \{0\}\}.$$

Position form:

$$Q^{\mu\nu}(\eta; t, y) := g_{\alpha\beta}(x^*(\eta))\, q^{\alpha\mu}(\eta; t, y)\, q^{\beta\nu}(\eta; t, y)\,,$$

where

$$q^{\alpha\mu}(\eta; t, y) := [x^*(\eta)]^{\alpha}_{\eta_\mu}\,.$$

Momentum form:

$$P^{\mu\nu}(\eta; t, y) := g_{\alpha\beta}(x^*(\eta))\, p^{\alpha\mu}(\eta; t, y)\, p^{\beta\nu}(\eta; t, y)\,,$$

where

$$p^{\alpha\mu}(\eta; t, y) := g^{\alpha\gamma}(x^*(\eta)) \left[(\xi^*_\gamma(\eta))_{\eta_\mu} - \Gamma^{\rho}{}_{\gamma\beta}(x^*(\eta))\, \xi^*_\rho(\eta)\, (x^*(\eta))^{\beta}_{\eta_\mu}\right].$$

Theorem 3 (Capoferri–Levitin–Vassiliev) Let a and b be positive parameters. Then the linear combination of the position and momentum forms

$$ah^2Q + bP$$

is a Riemannian metric on the Lagrangian manifold $\mathcal{T}_y(t)$.

Theorem 4 (Capoferri–Levitin–Vassiliev) We have

$$\varphi_{x^\alpha \eta_\mu}\Big|_{x=x^*} = g_{\alpha\beta}(x^*)\left[p^{\beta\mu} - i\,\epsilon\,h\,q^{\beta\mu}\right].$$

Recall that p and q are, effectively, square roots of P and Q.

Maslov index

Calculate the increment of

$$-\frac{1}{2\pi}\,\arg\det{}^2\varphi_{x^\alpha\eta_\beta}\Big|_{x=x^*}$$

along the closed geodesic.

For the 2-sphere

$$\det\varphi_{x^\alpha\eta_\beta}\Big|_{x=x^*}=[\rho(x)]\,[\rho(y)]^{-1}\,[\cos(t)-i\,\epsilon\,\sin(t)]\,,$$

which gives Maslov index 2.

Future plans

- Handle the massless Dirac operator on a 3-manifold.
- Handle the operator curl on a 3-manifold.
- Attempt to develop a fully relativistic version of our method.