

# Spacetime diffeomorphisms as matter fields

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# Playing field

Let  $(M, g)$  be a Lorentzian 4-manifold.

Consider a diffeomorphism  $\varphi : M \rightarrow M$ . This is the unknown quantity of our mathematical model.

Second Lorentzian metric  $h := \varphi^* g$ , the pullback of  $g$ .

A pair of metrics,  $g$  and  $h$ , allows us to write down an action (variational functional).

## Strain tensor

Linear algebra: a pair of non-degenerate symmetric bilinear forms  $g, h : V \times V \rightarrow \mathbb{R}$  in a real finite-dimensional vector space  $V$  defines an invertible linear operator  $L : V \rightarrow V$  via the formula

$$h(u, v) = g(Lu, v), \quad \forall u, v \in V.$$

Convenient to subtract the identity operator,

$$S := L - \text{Id}$$

Definition of strain tensor:

$$S^\alpha{}_\beta(x) := [g^{\alpha\gamma}(x)] [h_{\gamma\beta}(x)] - \delta^\alpha{}_\beta.$$

Describes, pointwise, linear map in the fibres of the tangent bundle

$$v^\alpha \mapsto S^\alpha{}_\beta v^\beta.$$

## Scalar invariants of the strain tensor

Obvious choice:  $\text{tr}(S^k)$ ,  $k = 1, 2, 3, 4$ .

More convenient choice:

$$e_1(\varphi) := \text{tr } S = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,$$

$$e_2(\varphi) := \frac{1}{2} [(\text{tr } S)^2 - \text{tr}(S^2)] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4,$$

$$e_3(\varphi) := \text{tr adj } S = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4,$$

$$e_4(\varphi) := \det S = \lambda_1\lambda_2\lambda_3\lambda_4.$$

Elementary symmetric polynomials. The  $\lambda_j$  are eigenvalues of  $S$ .

**Remark 1** In the Lorentzian setting strain may be nilpotent with nilpotency index  $\leq 3$  (Gohberg, Lancaster and Rodman 2005). This means that a Lorentzian spacetime can be deformed in a most unusual way: strain is nonzero but all the scalar invariants are zero.

# Action

$$\mathcal{J}(\varphi) := \int_M \mathcal{L}(e_1(\varphi), e_2(\varphi), e_3(\varphi), e_4(\varphi)) \sqrt{-\det g_{\mu\nu}(x)} dx,$$

where  $\mathcal{L}$  is some prescribed smooth real-valued function of four real variables and  $dx := dx^1 dx^2 dx^3 dx^4$ .

# Examples of meaningful Lagrangians

**Example 1** Lagrangian linear in strain

$$\mathcal{L}(e_1, e_2, e_3, e_4) = e_1.$$

This is the Lagrangian of a harmonic map. No free parameters.

**Example 2** Lagrangian quadratic in strain

$$\mathcal{L}(e_1, e_2, e_3, e_4) = \alpha(e_1)^2 + \beta e_2,$$

where  $\beta \neq 0$ . This is the Lagrangian of an elastic continuum that is physically linear. One free parameter:  $\frac{\alpha}{\beta}$ .

Poisson's ratio:  $\nu = \frac{2\alpha + \beta}{4\alpha + \beta}$ .

# Our mathematical model

Vary action subject to the volume preservation constraint

$$\det g_{\alpha\beta}(x) = \det h_{\mu\nu}(x).$$

Work in the subgroup of volume-preserving diffeomorphisms.

Equivalent representation of the volume preservation constraint:

$$e_1(\varphi) + e_2(\varphi) + e_3(\varphi) + e_4(\varphi) = 0.$$

## Conditions on the Lagrangian $\mathcal{L}(e_1, e_2, e_3, e_4)$

Put  $f(z) := \mathcal{L}(-z, z, 0, 0)$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

**Condition 1**  $f'(0) \neq 0$ .

**Condition 2**  $f'(c) = 0$  for some  $c > 0$ .



## Our field equations

Introduce a Lagrange multiplier  $p : M \rightarrow \mathbb{R}$ . Can be interpreted as pressure, like in Stokes equations.

Our field equations read

$$E(\varphi) - dp = 0, \quad (1)$$

where  $E$  is a nonlinear second order partial differential operator mapping a diffeomorphism to a covector field and  $dp$  is the gradient of pressure  $p$ .

Equation (1) has to be complemented by the volume preservation constraint. We get a system of five partial differential equations for five unknowns: four components of  $\varphi$  and  $p$ .

## Vector field of displacements $A : M \rightarrow TM$

Connect a point  $P \in M$  with the point  $\varphi(P) \in M$  by a geodesic  $\gamma : [0, 1] \rightarrow M$ , so that  $\gamma(0) = P$  and  $\gamma(1) = \varphi(P)$ . Parameterize the geodesic in such a way that  $\gamma(\tau)$  is a solution of the equation

$$\ddot{\gamma}^\lambda + \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} \dot{\gamma}^\mu \dot{\gamma}^\nu = 0,$$

where the dot stands for differentiation in  $\tau$ .

Then

$$A(P) := \dot{\gamma}(0).$$

# Linearised field equations

**Theorem 1** Our linearised field equations read

$$\begin{pmatrix} \delta d - 2 \operatorname{Ric} & d \\ \delta & 0 \end{pmatrix} \begin{pmatrix} A^b \\ \rho \end{pmatrix} = 0. \quad (2)$$

**Remark 2** System (2) has Agmon–Douglis–Nirenberg structure.

**Remark 3** For Ricci-flat manifolds we get Maxwell's equations

$$\delta d A^b = J$$

in the Lorenz gauge

$$\delta A^b = 0$$

with exact current  $J := -d\rho$ .

# Solving nonlinear equations: group-theoretic trick

**Definition 1** Let  $\varphi$  be a diffeomorphism. We say that  $\varphi$  is *homogeneous* if there exists a subgroup  $H$  of the isometry group acting transitively on  $M$  and satisfying

$$H \circ \varphi = \varphi \circ H.$$

If we have the stronger property

$$\xi \circ \varphi = \varphi \circ \xi, \quad \forall \xi \in H,$$

we say that  $\varphi$  is *equivariant*.

**Theorem 2** Let  $\varphi$  be a homogeneous diffeomorphism. Then all the scalar invariants are constant. Furthermore, if the equation  $E(\varphi) = 0$  is satisfied at a single point then it is satisfied on the whole manifold.

# Minkowski space

Work in Minkowski space  $\mathbb{M}$ ,  $g_{\alpha\beta} = \text{diag}(1, 1, 1, -1)$ .

10-dimensional isometry group of  $\mathbb{M}$ , the Poincaré group.

Restricted Poincaré group (identity component of Poincaré group).

Need to find nontrivial 4-dimensional subgroups of the restricted Poincaré group.

# Massless screw groups, right-handed and left-handed

This is the set of isometries

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \mapsto \begin{pmatrix} \cos(q^3 + q^4) & \mp \sin(q^3 + q^4) & 0 & 0 \\ \pm \sin(q^3 + q^4) & \cos(q^3 + q^4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} + \begin{pmatrix} q^1 \\ q^2 \\ q^3 \\ q^4 \end{pmatrix}, \quad q \in \mathbb{R}^4.$$

## Massive screw group

For a given value of parameter  $m > 0$ , this is the set of isometries

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \mapsto \begin{pmatrix} \cos(2mq^4) & -\sin(2mq^4) & 0 & 0 \\ \sin(2mq^4) & \cos(2mq^4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} + \begin{pmatrix} q^1 \\ q^2 \\ q^3 \\ q^4 \end{pmatrix}, \quad q \in \mathbb{R}^4.$$

# Explicit massless solutions of our nonlinear field equations

**Theorem 3** Let  $a > 0$  be parameter. Then the diffeomorphism described by the vector field of displacements

$$A^\alpha(x) = a \begin{pmatrix} \cos(x^3 + x^4) \\ \pm \sin(x^3 + x^4) \\ 0 \\ 0 \end{pmatrix}$$

is volume preserving and satisfies our nonlinear field equations.



# Explicit massive solutions of our nonlinear field equations

**Theorem 4** Let  $m > 0$ ,  $a > 0$  and  $b \in \mathbb{R}$  be parameters satisfying

$$4m^2(a^2 + b^2) = c,$$

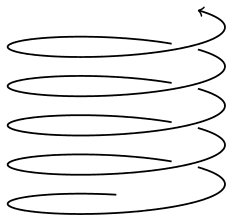
where  $c$  is the critical point from an earlier slide. Then the diffeomorphism described by the vector field of displacements

$$A^\alpha(x) = \begin{pmatrix} a \cos(2mx^4) \\ a \sin(2mx^4) \\ 2mbx^4 \\ 0 \end{pmatrix}$$

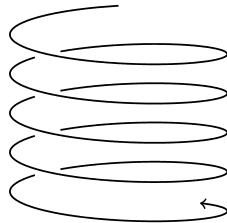
is volume preserving and satisfies our nonlinear field equations.

**Remark 4** We have here two free parameters,  $m > 0$  and  $b \in \mathbb{R}$ .

- ▶  $m$  is quantum mechanical mass.
- ▶  $b$  may be interpreted as electric charge.



(i)  $b > 0$



(ii)  $b < 0$

Figure 1: Massive solution

# The Dirac equation

**Claim** One can see the Dirac equation in our explicit solutions.

Need to separate stretches and rotations.

Deformation gradient

$$D^{\alpha}_{\beta} := \delta^{\alpha}_{\beta} + \frac{\partial A^{\alpha}}{\partial x^{\beta}}.$$

It contains more geometric information than the strain tensor.

Polar decomposition

$$D^{\alpha}_{\beta} = U^{\alpha}_{\gamma} V^{\gamma}_{\beta},$$

where  $U$  is Lorentz-orthogonal and  $V$  is Lorentz-symmetric.

Define the rotation 2-form as  $F := \ln U$ .

For our explicit solutions the rotation 2-form  $F$  admits a natural complexification  $\mathbb{F}$ ,

$$F = \operatorname{Re} \mathbb{F}.$$

Turns out, this complex-valued 2-form is degenerate,

$$\det \mathbb{F} = \det(*\mathbb{F}) = 0,$$

therefore it is equivalent to the square of a bispinor field.

Turns out, this bispinor field satisfies the Dirac equation.