Yuri's influence on my work and my life

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$\approx$ 1983 met Yuri in Moscow. Yuri is PhD student, finishing thesis.

First joint paper

Yu. Safarov and D. Vassiliev, Branching Hamiltonian billiards, *Soviet Mathematics Doklady* **38** (1989), no. 1, 64–68.

Definition A Hamiltonian is a positive function $h(x,\xi)$ on $T^*M \setminus \{0\}$ positively homogeneous in momentum ξ of degree 1.

$$\dot{x} = h_\xi(x,\xi), \qquad \dot{\xi} = -h_x(x,\xi),$$

Reflection law:

- $x(t)$ is continuous,
- component of $\xi(t)$ tangent to boundary is continuous and
- the value of the Hamiltonian is preserved.

Why branching? Because

- the Hamiltonian is not necessarily the square root of a quadratic form in momentum ξ and
- there may be several Hamiltonians.

$\approx$ 1988 Yuri starts visiting Linköping (Sweden) at the invitation of Ari. This has a profound effect on his view of life. Yuri happily shares his views with everyone.

$\approx$ 1989 Simon Gindikin (Rutgers) visits Moscow and suggests that Yuri and I write a book.

We sign contracts with the AMS and Nauka.

Spring 1990: a week spent at country retreat for communist party officials.

1989: Yuri's habilitation (DSc thesis).

Second Weyl coefficient for first order systems

A is a first order elliptic self-adjoint matrix operator acting on a closed manifold without boundary. $A_1(x,\xi)$ and $A_{\mathrm{sub}}(x,\xi)$ are its matrix-valued principal and subprincipal symbols.

What is the structure of the second Weyl coefficient?

J.J.Duistermaat and V.W.Guillemin (1975): in the scalar case structure of the second Weyl coefficient is determined by A_{sub}.

Early papers of V.Ivrii and G.Rozenblum: replace A_{sub} with

$$[v^{(j)}]^* A_{\mathrm{sub}} v^{(j)}.$$

Yu.Safarov (1989):

$$[v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)} I, v^{(j)}\}.$$

O.Chervova, R.J.Downes and D.Vassiliev (2013):

$$[v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)} I, v^{(j)} + \frac{i}{n-1} h^{(j)}\{[v^{(j)}]^*, v^{(j)}\}.$$

The circle group $\mathrm{U}(1)$

$$\mathrm{U}(1) = \{z \in \mathbb{C} : \ |z| = 1\}.$$

Here the group operation is multiplication.

Why is the circle group $\mathrm{U}(1)$ relevant?

Let $v^{(j)}(x,\xi)$ be an eigenvector of the principal symbol.

Problem: $v^{(j)}(x,\xi)$ is not defined uniquely. It is is defined modulo a gauge transformation

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)}$$

where

$$\phi^{(j)} : T^*M \setminus \{0\} \to \mathbb{R}$$

is an arbitrary smooth function. This gives rise to a $\mathrm{U}(1)$ connection which, in turn, generates curvature.

Physical meaning of the $\mathrm{U}(1)$ connection

In theoretical physics a $\mathrm{U}(1)$ connection is usually associated with electromagnetism. The corresponding curvature tensor is the electromagnetic (Faraday) tensor.

I have shown that inside **any** system of partial differential equations with variable coefficients there is an intrinsic electromagnetic field which lives on the cotangent bundle. Abstract mathematical fact.

One has to take account of this intrinsic electromagnetic field in order to get correct results.

My latest paper on the subject

Z.Avetisyan, J.Sjöstrand and D.Vassiliev, The second Weyl coefficient for a first order system. Preprint arXiv:1801.00757.

Another strand: Maxwell's equations

$\approx$ 1990: I serve as an external referee on Gureev's PhD thesis.

The cases of a domain in $\mathbb{R}^3$ and a Riemannian 3-manifold are completely different. On a Riemannian 3-manifold can perform complexification

$$w := E + iH$$

to get

$$\frac{\partial w}{\partial t} + \operatorname{curl} w = 0.$$

End of 1992 (?): Yuri comes to the UK on a 6 month (?) Royal Society grant.

1993: permanent appointment at King's.


Acorn MOS
BASIC
ROM F TERMINAL 01
ROM E VIEW 04 unplugged
ROM D Acorn ADFS 53
ROM C BASIC 04
ROM B Edit 01
ROM A ViewSheet 02
ROM 9 DFS 79
ROM 5 DOOMSDAY 00

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Global invariant construction of the propagator

For simplicity, let A be a first order elliptic self-adjoint scalar operator acting on a closed manifold without boundary. Then

$$U(t) := e^{-itA}$$

is the *propagator*. It is a Fourier integral operator.

Yuri: the propagator can be written as a single integral, globally in space and in time and in coordinate-free notation.

A.Laptev, Yu.Safarov and D.Vassiliev. On global representation of Lagrangian distributions and solutions of hyperbolic equations, *Communications on Pure and Applied Mathematics* **47** (1994), no. 11, 1411–1456.

Also two preliminary versions of this paper published in 1992.

Global complex-valued phase function

Hamiltonian trajectories $(x^*(t;y,\eta),\xi^*(t;y,\eta))$.

Here $(y,\eta)\in T^*M\setminus\{0\}$ is the starting point.

Definition of global complex-valued phase function:

$$\varphi(t,x;y,\eta)=(x-x^*)^\alpha\,\xi^*_\alpha+O(|x-x^*|^2),$$

$$\operatorname{Im}\varphi\geq 0,$$

$$\det\varphi_{x^\alpha\eta_\beta}\Big|_{x=x^*}\neq 0.$$

Maslov index

Important quantity

$$\det \varphi_{x^\alpha \eta_\beta}\Big|_{x=x^*} \ .$$

$\approx$ 1994 talk at Donaldson's seminar at Oxford.

Subprincipal symbol of a propagator

In the presence of a Riemannian metric we can define the phase function in a canonical way

$$\varphi(t, x; y, \eta; \epsilon).$$

Already appears in the original 1994 paper.

Canonical choice of phase function allows us to to define the subprincipal symbol of a propagator.

Matteo Capoferri is studying this issue. Performed detailed analysis for the Laplace–Beltrami operator on a 2-sphere.

We will use this construction in a series of papers dealing with Laplace–Beltrami, massless Dirac and curl.