Rotational elasticity

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Describing a 3-dimensional elastic medium

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity: displacements and rotations. See E.Cosserat and F.Cosserat, *Théorie des Corps Déformables*, 1909. Reprinted by Cornell and now available from Amazon.

(c) Rotational elasticity: rotations only.

Classical and rotational elasticity are two limit cases of Cosserat.

Motivation for rotational elasticity.

(a) Curiosity.

(b) MacCullagh, 1839. Tried modelling world aether in terms of rotational elasticity. Inadequate mathematical apparatus.

(c) A. Einstein and É. Cartan, 1920s. Teleparallelism = absolute parallelism = fernparallelismus. Formal geometric definition of teleparallelism: curvature is zero but torsion is nonzero. Compare with general relativity: torsion is zero but curvature is nonzero.

Note: Cartan knew the Cosserat book. He wrote that he drew inspiration from the 'beautiful' work of the Cosserat brothers.

(d) Ericksen fluid. See J. L. Ericksen, Twist waves in liquid crystals, *Q. Jl Mech. Appl. Math.* **21** (1968) 463–465.

Elastic medium occupies $\mathbb{R}^3$. To describe rotations of material points I attach to each geometric point of $\mathbb{R}^3$ a *coframe*.

A coframe ϑ is a triple ϑ^j, $j = 1, 2, 3$, of orthonormal covector fields. Each ϑ^j has hidden tensor index: $\vartheta^j = \vartheta^j{}_\alpha$, $\alpha = 1, 2, 3$.

Same in plain English: a coframe is a field of orthonormal bases.

Can think of the coframe as a field of orthogonal matrices $\vartheta^j{}_\alpha$.

NB. Coframe lives separately from Cartesian coordinates. It is not aligned with coordinate lines.

The coframe ϑ is a dynamical variable (unknown quantity).

The other dynamical variable is a density ρ.

Measuring rotational deformations

The natural measure of rotational deformations is *torsion*

$$T := \vartheta^1 \otimes d\vartheta^1 + \vartheta^2 \otimes d\vartheta^2 + \vartheta^3 \otimes d\vartheta^3.$$

Torsion is a rank 3 tensor antisymmetric in the last pair of indices.

Irreducible decomposition of rotational deformations

$$T = T^{\mathrm{ax}} + T^{\mathrm{vec}} + T^{\mathrm{ten}}.$$

Adjectives *axial*, *vector* and *tensor*.

Vector torsion is sometimes called trace torsion.

Potential energy

$$P(t) = \int_{\mathbb{R}^3} \Big(c^{\mathrm{ax}} \|T^{\mathrm{ax}}\|^2 + c^{\mathrm{vec}} \|T^{\mathrm{vec}}\|^2 + c^{\mathrm{ten}} \|T^{\mathrm{ten}}\|^2 \Big) \rho \, dx^1 dx^2 dx^3 .$$

Kinetic energy

$$K(t) = c^{\mathrm{kin}} \int_{\mathbb{R}^3} \|\Omega\|^2 \rho \, dx^1 dx^2 dx^3 ,$$

where $\Omega = \frac{1}{2} {*}(\vartheta^1 \wedge \partial_t \vartheta^1 + \vartheta^2 \wedge \partial_t \vartheta^2 + \vartheta^3 \wedge \partial_t \vartheta^3)$ is the (pseudo)vector of angular velocity.

Action (variational functional) of rotational elasticity

$$S(\vartheta, \rho) = \int_{\mathbb{R}} \Big(P(t) - K(t) \Big) dt \, .$$

Euler–Lagrange equations: vary coframe ϑ and density ρ.

Model is physically linear but geometrically nonlinear.

Solving Euler–Lagrange equations

Varying coframe is difficult because of kinematic constraint: covectors ϑ^j, $j = 1, 2, 3$, have to remain orthonormal. Could use Euler angles (yaw, pitch, and roll) but this is inconvenient.

Most convenient description of rotations in $\mathbb{R}^3$: switch to spinors

coframe ϑ and positive density ρ

$$\Updownarrow$$

nonvanishing 2-component complex spinor field ξ modulo sign

Plane wave solutions

Look first for *plane wave* solutions

$$\xi(x^1, x^2, x^3, t) = e^{-i(k_1 x^1 + k_2 x^2 + k_3 x^3 + \omega t)} \zeta \,.$$

Theorem 1 *My Euler–Lagrange equations admit plane wave solutions with velocities* $\sqrt{\dfrac{4c^{\mathsf{ax}} + 2c^{\mathsf{ten}}}{3c^{\mathsf{kin}}}}$ *and* $\sqrt{\dfrac{c^{\mathsf{vec}} + c^{\mathsf{ten}}}{2c^{\mathsf{kin}}}}$ *.*

Purely axial material

$$c^{\mathrm{ax}} \neq 0, \qquad c^{\mathrm{vec}} = c^{\mathrm{ten}} = 0.$$

Potential energy feels only the axial deformation, i.e. only the totally antisymmetric piece of torsion.

Look for *stationary* solutions

$$\xi(x^1, x^2, x^3, t) = e^{-i\omega t}\eta(x^1, x^2, x^3).$$

Theorem 2 *In the stationary setting my Euler–Lagrange equations are equivalent to a pair of massless Dirac equations*

$$i\begin{pmatrix} \mp\partial_t + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \mp\partial_t - \partial_3 \end{pmatrix}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = 0.$$

Summary

- Cosserat elasticity is a fun subject.

- Adding rotational degrees of freedom to material points leads to the appearance of interesting new waves.

Papers and preprints can be found on my web page

`http://www.homepages.ucl.ac.uk/~ucahdva/`

My talks (including this one) are also on my web page.