Gauge theory from an analyst's perspective

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Object of study

Let M be an n-manifold equipped with local coordinates $x = (x^1, \ldots, x^n)$.

Will work with m-columns $u : M \to \mathbb{C}^m$ of scalar fields.

First order Hermitian sesquilinear form:

$$S(u, v) := \int_M \left[-\frac{i}{2} v^* \mathbf{S}_1^\alpha \frac{\partial u}{\partial x^\alpha} + \frac{i}{2} \frac{\partial v^*}{\partial x^\alpha} \mathbf{S}_1^\alpha u + v^* \mathbf{S}_0 u \right] dx^1 \ldots dx^n,$$

where $\mathbf{S}_1^\alpha$ is a matrix-valued vector density and $\mathbf{S}_0$ is a matrix-valued density. These matrix-functions are assumed to be Hermitian: $(\mathbf{S}_1^\alpha)^* = \mathbf{S}_1^\alpha$, $\mathbf{S}_0^* = \mathbf{S}_0$.

I reserve bold font for density-valued quantities.

My sesquilinear form generates a linear first order Euler–Lagrange equation $Lu = 0$.

Non-degeneracy condition

Define the density-valued principal symbol

$$\mathbf{L}_{\mathrm{prin}}(x,p) := \mathbf{S}_1^{\alpha}(x)\, p_\alpha \, .$$

Here $p = (p_1, \dots, p_n)$ is the dual variable (momentum).

We say that our sesquilinear form S is *non-degenerate* if

$$\mathbf{L}_{\mathrm{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}.$$

Gauge transformations of sesquilinear forms

Definition 1 Sesquilinear forms S and $\tilde{S}$ are said to be GL-equivalent if there exists a smooth matrix-function $Q : M \to GL(m, \mathbb{C})$ such that $\tilde{S}(u, v) = S(Qu, Qv)$.

Definition 2 Sesquilinear forms S and $\tilde{S}$ are said to be GL^+-equivalent if there exists a smooth matrix-function $Q : M \to GL(m, \mathbb{C})$ with $\det Q > 0$ s. t. $\tilde{S}(u, v) = S(Qu, Qv)$.

I view the map $u \mapsto Qu$ as a gauge transformation.

Task: write down necessary and sufficient conditions for a pair of sesquilinear forms to be equivalent. I want to describe explicitly equivalence classes of sesquilinear forms.

Special case: two scalar fields over a 4-manifold

The case $n = 4$, $m = 2$ is special.

Can provide complete description of equivalence classes of sesquilinear forms.

Main results in

Z. Avetisyan, Y.-L. Fang, N. Saveliev and D. Vassiliev, *Analytic definition of spin structure*, JMP **58** (2017) 082301.

Lorentzian metric appears out of thin air

The determinant of the principal symbol is a quadratic form in momentum

$$\det \mathbf{L}_{\mathrm{prin}}(x,p) = -\mathbf{g}^{\alpha\beta}(x)\, p_\alpha p_\beta \,,$$

where $\mathbf{g}^{\alpha\beta}(x)$ is a real symmetric 4×4 matrix-function with values in 2-densities.

Lemma 1 The matrix-function $\mathbf{g}^{\alpha\beta}(x)$ has Lorentzian signature, i.e. it has three positive eigenvalues and one negative eigenvalue.

My definition of the metric tensor:

$$g^{\alpha\beta}(x) := |\det \mathbf{g}^{\mu\nu}(x)|^{-1/3}\, \mathbf{g}^{\alpha\beta}(x)\,.$$

Note that my gauge transformations preserve the conformal class of metrics.

Other geometric objects encoded within a sesquilinear form

- Orthonormal frame ${e_j}^\alpha$. Behaves covariantly under $GL(2,\mathbb{C})$ transformations.
- Electromagnetic covector potential A. Invariant under $GL^+(2,\mathbb{C})$ transformations. **Not** invariant under $GL(2,\mathbb{C})$ transformations: picks up the gradient of a real-valued scalar field.
- Electromagnetic tensor dA. Invariant under $GL(2,\mathbb{C})$ transformations.

Explicit formula for the orthonormal frame

Introduce 'normal', as opposed to density-valued, principal symbol

$$L_{\mathrm{prin}}(x,p) := |\det g_{\mu\nu}(x)|^{-1/2}\,\mathbf{L}_{\mathrm{prin}}(x,p)\,.$$

It is an invariantly defined 2×2 Hermitian matrix-function on T^*M.

Decomposing the principal symbol with respect to the standard basis

$$s^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad s^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad s^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad s^4=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

in the real vector space of 2×2 Hermitian matrices, we get

$$L_{\mathrm{prin}}(x,p)=s^j\,{e_j}^\alpha(x)\,p_\alpha\,.$$

Explicit formula for the electromagnetic covector potential

Pretty complicated.

It involves the generalised Poisson bracket on matrix-functions

$$\{U, V, W\} := U_{x^\alpha} V\, W_{p_\alpha} - U_{p_\alpha} V\, W_{x^\alpha}$$

and the operator of matrix adjugation

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \operatorname{adj} U$$

from elementary linear algebra.

Why is the formula for the electromagnetic covector potential complicated? Because it implicitly contains connection coefficients for spinor fields.

Classification of sesquilinear forms in the GL^+ guage

Theorem 1 A pair of sesquilinear forms S and $\tilde{S}$ is GL^+-equivalent iff

- their metrics are in the same conformal class,
- their electromagnetic covector potentials are the same,
- their topological charges are the same,
- their temporal charges are the same and
- they have the same spin structure.

Here I use the following analytic definition of spin structure.

Definition 3 Principal symbols L_{prin} and $\tilde{L}_{\mathrm{prin}}$ are said to be GL^+-equivalent if there exists a smooth matrix-function $Q : M \to GL(2, \mathbb{C})$ with $\det Q > 0$ such that $\tilde{L}_{\mathrm{prin}} = Q^* L_{\mathrm{prin}} Q$. An equivalence class of principal symbols is called spin structure.

Classification of sesquilinear forms in the GL guage

Theorem 2 A pair of sesquilinear forms S and $\tilde{S}$ is GL-equivalent iff

- their metrics are in the same conformal class,
- their electromagnetic tensors are the same,
- their topological charges are the same,
- their temporal charges are the same and
- some topological conditions are satisfied.

Here the topological conditions are weaker than those for the GL^+ gauge. Something weaker than the spin structure condition.

Bottom line

Suppose that I am looking at a system of two linear first order PDEs for two unknown complex-valued scalar fields over a 4-manifold.

Suppose that I know that this system of PDEs admits a variational formulation.

Then Lorentzian geometry is automatically encoded within this system of PDEs.

There is no need to introduce geometric constructs a priori. They are already there.