An analyst's take on gauge theory

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Talk is based on the paper

Z. Avetisyan, Y.-L. Fang, N. Saveliev and D. Vassiliev, *Analytic definition of spin structure*, JMP **58** (2017) 082301.

Object of study

Let M be a connected m-dimensional manifold without boundary, local coordinates $x=(x^1,\dots,x^m)$.

Will work with n-columns $u : M \to \mathbb{C}^n$ of scalar fields.

First order sesquilinear form:

$$S(u,v) := \int_M \left[u^* \mathbf{A}^\alpha \left(\frac{\partial v}{\partial x^\alpha} \right) + \left(\frac{\partial u}{\partial x^\alpha} \right)^* \mathbf{B}^\alpha v + u^* \mathbf{C} v \right] dx \,,$$

where $\mathbf{A}^\alpha$ and $\mathbf{B}^\alpha$ are matrix-valued vector densities, $\mathbf{C}$ is a matrix-valued density and $dx = dx^1 \dots dx^m$.

I reserve bold font for density-valued quantities.

The symbol of a first order sesquilinear form

Canonical representation of a first order sesquilinear form:

$$S(u,v)=\int_M \left[-\frac{i}{2}u^*\mathbf{E}^\alpha\left(\frac{\partial v}{\partial x^\alpha}\right)+\frac{i}{2}\left(\frac{\partial u}{\partial x^\alpha}\right)^*\mathbf{E}^\alpha v+u^*\mathbf{F}v\right]dx\,.$$

Density-valued principal symbol $\mathbf{S}_{\mathrm{prin}}(x,p):=\mathbf{E}^\alpha(x)\,p_\alpha\,.$

Density-valued subprincipal symbol $\mathbf{S}_{\mathrm{sub}}(x):=\mathbf{F}(x)\,.$

Density-valued full symbol $\mathbf{S}_{\mathrm{full}}(x,p):=\mathbf{S}_{\mathrm{prin}}(x,p)+\mathbf{S}_{\mathrm{sub}}(x)\,.$

Full symbol uniquely determines the sesquilinear form.

Sesquilinear form is Hermitian iff its full symbol is Hermitian.

Non-degeneracy condition

Definition 1 We say that our Hermitian first order sesquilinear form S is *non-degenerate* if

$$\mathbf{S}_{\mathrm{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}.$$

Gauge transformations of sesquilinear forms

Definition 2 We say that two sesquilinear forms, S and $\tilde{S}$, are *GL-equivalent* if

$$\tilde{S}(u, v) = S(Ru, Rv) \tag{1}$$

for some smooth matrix-function

$$R : M \to GL(n, \mathbb{C}). \tag{2}$$

Definition 3 We say that two sesquilinear forms, S and $\tilde{S}$, are *SL-equivalent* if we have (1) for some smooth matrix-function

$$R : M \to SL(n, \mathbb{C}). \tag{3}$$

I view the map $u \mapsto Ru$ as a gauge transformation.

Task at hand

Give explicit necessary and sufficient conditions for a pair of non-degenerate Hermitian first order sesquilinear forms to be GL-equivalent or SL-equivalent.

I want to describe equivalence classes of sesquilinear forms.

Special case: two scalar fields over a 4-manifold

The case $m = 4$, $n = 2$ is special.

We can provide a complete description of equivalence classes of sesquilinear forms, both in the GL setting and the SL setting.

Lorentzian metric appears out of thin air

The determinant of the principal symbol is a quadratic form in momentum

$$\det \mathbf{L}_{\mathrm{prin}}(x,p) = -\mathbf{g}^{\alpha\beta}(x)\, p_\alpha p_\beta \,,$$

where $\mathbf{g}^{\alpha\beta}(x)$ is a real symmetric 4×4 matrix-function with values in 2-densities.

Lemma 1 The matrix-function $\mathbf{g}^{\alpha\beta}(x)$ has Lorentzian signature, i.e. it has three positive eigenvalues and one negative eigenvalue.

My definition of the metric tensor:

$$g^{\alpha\beta}(x) := |\det \mathbf{g}^{\mu\nu}(x)|^{-1/3}\, \mathbf{g}^{\alpha\beta}(x)\,.$$

Invariant under $SL(2,\mathbb{C})$ transformations. Not invariant under $GL(2,\mathbb{C})$ transformations but conformal class is preserved.

Other geometric objects encoded within a sesquilinear form

- Orthonormal frame Covariant under $GL(2,\mathbb{C})$ transformations.
- Electromagnetic covector potential A. Invariant under $SL(2,\mathbb{C})$ transformations. Not invariant under $GL(2,\mathbb{C})$ transformations: picks up the gradient of a scalar field.
- Electromagnetic tensor dA. Invariant under $GL(2,\mathbb{C})$ transformations.
- Spin structure.

Definition 4 We say that two principal symbols, $\mathbf{S}_{\mathrm{prin}}$ and $\tilde{\mathbf{S}}_{\mathrm{prin}}$, are *equivalent* if we have

$$\tilde{\mathbf{S}}_{\mathrm{prin}} = R^* \mathbf{S}_{\mathrm{prin}} R$$

for some smooth matrix-function $R : M \to SL(2,\mathbb{C})$. An equivalence class of principal symbols is called *spin structure*.

Bottom line, in plain English

Suppose that I am looking at a system of two linear first order PDEs for two unknown complex-valued scalar fields over a 4-manifold.

Suppose that I know that this system of PDEs admits a variational formulation.

Then Lorentzian geometry is automatically encoded within this system of PDEs.

There is no need to introduce geometric constructs a priori. They are already there.