First order systems of PDEs on manifolds without boundary: understanding neutrinos and photons

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Why this talk is different

1. I do not have publications on Maxwell's equations (yet).

2. I work on a closed manifold, not a domain in Euclidean space.

3. I am motivated by particle physics.

Playing field

Let M be a closed n-dimensional manifold, $n \geq 2$. Will denote local coordinates by $x = (x^1, \ldots, x^n)$.

A half-density is a quantity $M \to \mathbb{C}$ which under changes of local coordinates transforms as the square root of a density.

Will work with m-columns $v : M \to \mathbb{C}^m$ of half-densities.

Inner product $\langle v, w \rangle := \int_M w^* v \, dx$, where $dx = dx^1 \ldots dx^n$.

Want to study a formally self-adjoint first order linear differential operator L acting on m-columns of complex-valued half-densities.

Need an invariant analytic description of my differential operator.

In local coordinates my operator reads

$$L = F^\alpha(x)\frac{\partial}{\partial x^\alpha} + G(x),$$

where $F^\alpha(x)$ and $G(x)$ are some $m \times m$ matrix-functions.

The principal and subprincipal symbols are defined as

$$L_{\mathsf{prin}}(x,p) := iF^\alpha(x)\,p_\alpha,$$

$$L_{\mathsf{sub}}(x) := G(x) + \frac{i}{2}(L_{\mathsf{prin}})_{x^\alpha p_\alpha}(x),$$

where $p = (p_1, \dots, p_n)$ is the dual variable (momentum).

Fact: L_{prin} and L_{sub} are invariantly defined Hermitian matrix-functions on T^*M and M respectively.

Fact: L_{prin} and L_{sub} uniquely determine the operator L.

We assume that our operator L is elliptic:

$$\det L_{\mathsf{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}.$$

Spectrum of L is discrete and accumulates to $+\infty$ and $-\infty$.

Spectral asymmetry: spectrum asymmetric about zero.

Technical assumption: $L_{\mathsf{prin}}(x,p)$ **has simple eigenvalues**

1. Without this assumption analysis is too difficult.

2. Even with this assumption analysis is difficult enough.

3. Most physically motivated problems satisfy this assumption.

First object of study: propagator

Let $x^{n+1} \in \mathbb{R}$ be the additional 'time' coordinate. Consider the Cauchy problem

$$w|_{x^{n+1}=0} = v \tag{1}$$

for the hyperbolic system

$$(-i\partial/\partial x^{n+1} + L)w = 0 \tag{2}$$

on $M \times \mathbb{R}$. The m-column of half-densities $v = v(x^1, \ldots, x^n)$ is given and the m-column of half-densities $w = w(x^1, \ldots, x^n, x^{n+1})$ is to be found. The solution of the Cauchy problem (1), (2) can be written as $w = U(x^{n+1})\,v$, where $U(x^{n+1})$ is the *propagator*.

Task: construct the propagator explicitly, modulo C^∞. Here "explicitly" means "reducing problem to solving ODEs".

Second object of study: the two counting functions

The two counting functions $N_\pm(\lambda) : (0, +\infty) \to \mathbb{N}$ are defined as

$N_+(\lambda) :=$ number of eigenvalues of operator L in interval $(0, \lambda)$,

$N_-(\lambda) :=$ number of eigenvalues of operator L in interval $(-\lambda, 0)$.

Task: derive asymptotic expansions

$$N_\pm(\lambda) = a_\pm \lambda^n + b_\pm \lambda^{n-1} + \dots$$

as $\lambda \to +\infty$, where $a_\pm$, $b_\pm$, ... are some real constants. Want explicit formulae for the Weyl coefficients $a_\pm$, $b_\pm$, ...

Third object of study: the eta function

The eta function of our operator L is defined as

$$\eta(s) := \sum_{\lambda_k \neq 0} \frac{\operatorname{sgn} \lambda_k}{|\lambda_k|^s} = \int_0^{+\infty} \lambda^{-s} \left(N'_+(\lambda) - N'_-(\lambda)\right) d\lambda \,,$$

where summation is carried out over all nonzero eigenvalues λ_k of our operator L and $s \in \mathbb{C}$ is the independent variable. The eta function is meromorphic in $\mathbb{C}$ with simple poles which can only occur at real integer values of s. No pole at $s = 0$.

The eta function is a measure of the asymmetry of the spectrum.

Task: evaluate the residues of $\eta(s)$.

Task: evaluate $\eta(0)$ (this is the so-called *eta invariant*).

Evaluating the second Weyl coefficient $b_\pm$ is not easy

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.

4 V.Ivrii, 1984, Springer Lecture Notes.

5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

6 V.Ivrii, book, 1998, Springer.

7 W.J.Nicoll, PhD thesis, 1998, University of Sussex.

8 I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.

9 O.Chervova, R.J.Downes and D.Vassiliev, 2013, Journal of Spectral Theory.

The U(1) connection

Each eigenvector $v^{(j)}(x,p)$, $j = 1,\ldots,m$, of the $m\times m$ matrix-function $L_{\mathsf{prin}}(x,p)$ is defined modulo a gauge transformation

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)},$$

where

$$\phi^{(j)} : T^*M \setminus \{0\} \to \mathbb{R}$$

is an arbitrary smooth real-valued function. There is a connection associated with this gauge degree of freedom, a U(1) connection on the cotangent bundle (similar to electromagnetism).

The U(1) connection has curvature, and this curvature appears in asymptotic formulae for the counting function and propagator.

Is my formula for the second Weyl coefficient $b_\pm$ correct?

Test: invariance under gauge transformations of the operator

$$L \mapsto R^* L R,$$

where

$$R : M \to \mathsf{U}(m)$$

is an arbitrary smooth unitary matrix-function.

Two by two operators are special

If $m = 2$ then $\det L_{\mathsf{prin}}$ is a quadratic form in momentum

$$\det L_{\mathsf{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta \,.$$

The coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, \ldots, n$, can be interpreted as components of a (contravariant) metric tensor.

Further on we always assume that $m = 2$.

Dimensions 2, 3 and 4 are special

Lemma 1 If $n \geq 5$, then our metric is degenerate, i.e.

$$\det g^{\alpha\beta}(x) = 0, \qquad \forall x \in M.$$

Further on we always assume that $n \leq 4$.

Dimensions 2, 3 and are even more special

Lemma 2 If $n = 4$, then our 2×2 operator L cannot be elliptic.

Further on we always assume that $n = 3$. This is the highest dimension in which one can have an elliptic 2×2 first order self-adjoint linear differential operator.

Additional assumption:

$$\mathrm{tr}\, L_{\mathrm{prin}}(x,p) = 0. \tag{3}$$

Logic: want to single out the simplest class of first order systems, expect to extract more geometry out of our asymptotic analysis and hope to simplify the results.

Lemma 3 Under the assumption (3) our metric is Riemannian, i.e. the metric tensor $g^{\alpha\beta}(x)$ is positive definite.

Note: half-densities are now equivalent to scalars. Just multiply or divide by $(\det g_{\alpha\beta}(x))^{1/4}$.

Extracting more geometry from our differential operator

Let us perform gauge transformations of the operator

$$L \mapsto R^* L R$$

where

$$R : M \to \mathrm{SU}(2)$$

is an arbitrary smooth special unitary matrix-function. Why unitary? Because I want to preserve the spectrum of my operator.

Principal and subprincipal symbols transform as

$$L_{\mathrm{prin}} \mapsto R^* L_{\mathrm{prin}} R,$$

$$L_{\mathrm{sub}} \mapsto R^* L_{\mathrm{sub}} R + \frac{i}{2}\left(R^*_{x^\alpha}(L_{\mathrm{prin}})_{p_\alpha} R - R^*(L_{\mathrm{prin}})_{p_\alpha} R_{x^\alpha}\right).$$

Problem: subprincipal symbol does not transform covariantly.

Solution: define *covariant* subprincipal symbol $L_{\text{csub}}(x)$ as

$$L_{\text{csub}} := L_{\text{sub}} - \frac{i}{16}\, g_{\alpha\beta}\{L_{\text{prin}}, L_{\text{prin}}, L_{\text{prin}}\}_{p_\alpha p_\beta}\,,$$

where subscripts p_α and p_β indicate partial derivatives and curly brackets denote the generalised Poisson bracket on matrix-functions

$$\{P, Q, R\} := P_{x^\alpha} Q R_{p_\alpha} - P_{p_\alpha} Q R_{x^\alpha}\,.$$

Electromagnetic covector potential appears out of thin air

Covariant subprincipal symbol can be uniquely represented as

$$L_{\text{csub}}(x) = L_{\text{prin}}(x, A(x)) + I A_4(x),$$

where $A = (A_1, A_2, A_3)$ is some real-valued covector field (magnetic covector potential), A_4 is some real-valued scalar field (electric potential) and I is the 2×2 identity matrix.

Geometric meaning of asymptotic coefficients

$$a_\pm = \frac{1}{6\pi^2} \int_M \sqrt{\det g_{\alpha\beta}}\; dx\,,$$

$$b_\pm = \mp \frac{1}{2\pi^2} \int_M A_4 \sqrt{\det g_{\alpha\beta}}\; dx\,.$$

Massless Dirac operator

Special case of the above construction, when electromagnetic potential is zero. Massless Dirac is determined by metric and spin structure modulo gauge transformations. Models neutrino.

- Geometers drop the adjective “massless”.

- “Massless Dirac” $\neq$ “Dirac type”.

- For massless Dirac the first **five** asymptotic coefficients of $N'_+(\lambda)$ and $N'_-(\lambda)$ are the same. Very difficult to observe spectral asymmetry for large λ.

- We studied spectral asymmetry for small λ.

- We found nontrivial families of metrics for which eigenvalues can be evaluated explicitly, both for the 3-torus and the 3-sphere.

Generalized Berger sphere

We work in $\mathbb{R}^4$ equipped with Cartesian coordinates (x^1, x^2, x^3, x^4). Consider the following three covector fields

$$e^1{}_\alpha = \begin{pmatrix} x^4 \\ x^3 \\ -x^2 \\ -x^1 \end{pmatrix}, \qquad e^2{}_\alpha = \begin{pmatrix} -x^3 \\ x^4 \\ x^1 \\ -x^2 \end{pmatrix}, \qquad e^3{}_\alpha = \begin{pmatrix} x^2 \\ -x^1 \\ x^4 \\ -x^3 \end{pmatrix}.$$

These covector fields are cotangent to the 3-sphere

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1.$$

We define the rank 2 tensor

$$g_{\alpha\beta} := \sum_{j,k=1}^{3} c_{jk}\, e^j{}_\alpha\, e^k{}_\beta$$

and restrict it to the 3-sphere. Here the c_{jk} are real constants, elements of a positive symmetric 3×3 matrix.


$\nabla \cdot D = \rho$
$\nabla \cdot B = 0$
$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \times H = \frac{\partial D}{\partial t} + J$

Maxwell's homogeneous vacuum equations on $M \times \mathbb{R}$:

$$\begin{pmatrix} \text{curl} & \partial/\partial x^4 \\ -\partial/\partial x^4 & \text{curl} \\ \text{div} & 0 \\ 0 & \text{div} \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix} = 0. \tag{4}$$

M is a closed oriented Riemannian 3-manifold. The operators curl and div act over M and can be written out explicitly using local coordinates (x^1, x^2, x^3) and the metric tensor.

$x^4 \in \mathbb{R}$ is the time coordinate.

Need to incorporate Maxwell's equations (4) into my scheme.

Step 1: complexification

Put $u := E + iB$. Then Maxwell's equations take the form

$$\begin{pmatrix} -i\partial/\partial x^4 + \text{curl} \\ \text{div} \end{pmatrix} u = 0.$$

Step 2: extension

$$\begin{pmatrix} -i\partial/\partial x^4 + \text{curl} & -\,\text{grad} \\ \text{div} & -i\partial/\partial x^4 \end{pmatrix} \begin{pmatrix} u \\ s \end{pmatrix} = 0.$$

Here s is an unknown complex-valued scalar field.

Extra eigenvalues coming from the Laplace-Beltrami operator.

Step 3: projection onto a frame

A *frame* is a triple of smooth orthonormal vector fields on M.

Topological fact: an oriented 3-manifold is parallelizable.

Hence, our oriented Riemannian 3-manifold M admits a frame.

After projection of the vector field u onto a frame extended Maxwell's equations take the form

$$(-i\partial/\partial x^4 + L)w = 0,$$

where w is a 4-column of complex-valued half-densities and L is a 4×4 elliptic self-adjoint first order linear differential operator.

Step 4: block diagonalization of principal symbol

Fact: there exists a linear transformation of our unknowns w which reduces extended Maxwell's equations to the form

$$\left[\begin{pmatrix} -i\partial/\partial x^4 + \text{Dirac} & 0 \\ 0 & -i\partial/\partial x^4 + \text{Dirac} \end{pmatrix} + \ 4 \times 4 \text{ matrix-function}\right] w = 0.$$