A non-geometric representation of the Dirac equation in curved spacetime

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Playing field

Let M be a 4-manifold, local coordinates $x=(x^1,x^2,x^3,x^4)$.

Let ρ be a prescribed positive density.

Will work with 2-columns $v:M\to\mathbb{C}^2$ of scalar fields.

Inner product $\langle v,w\rangle := \int_M w^*v\,\rho\,dx$, where $dx=dx^1dx^2dx^3dx^4$.

Want to study a formally self-adjoint first order linear differential operator L acting on 2-columns of complex-valued scalar fields.

Invariant analytic description of a differential operator

In local coordinates our operator reads

$$L = U^\alpha(x)\frac{\partial}{\partial x^\alpha} + V(x),$$

where $U^\alpha(x)$ and $V(x)$ are some 2×2 matrix-functions.

The principal and subprincipal symbols are defined as

$$L_{\mathrm{prin}}(x,p) := iU^\alpha(x)\, p_\alpha,$$

$$L_{\mathrm{sub}}(x) := V(x) + \frac{i}{2}(L_{\mathrm{prin}})_{x^\alpha p_\alpha}(x) + \frac{i}{2}L_{\mathrm{prin}}(x, \mathrm{grad}(\ln\rho(x))),$$

where $p = (p_1, p_2, p_3, p_4)$ is the dual variable (momentum).

Fact: L_{prin} and L_{sub} are invariantly defined 2×2 Hermitian matrix-functions on T^*M and M respectively.

Fact: L_{prin} and L_{sub} uniquely determine the operator L.

We say that our operator L is *non-degenerate* if

$$L_{\mathrm{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}.$$

Lorentzian metric appears out of thin air

The determinant of the principal symbol is a quadratic form in momentum

$$\det L_{\text{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta$$

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha,\beta = 1,2,3,4$, can be interpreted as components of a (contravariant) metric tensor.

Lemma Our metric is Lorentzian, i.e. the metric tensor $g^{\alpha\beta}(x)$ has three positive eigenvalues and one negative eigenvalue.

Extracting more geometry from my differential operator

Let us perform gauge transformations of the original operator

$$L \mapsto R^* L R$$

where

$$R : M \to \mathrm{SL}(2, \mathbb{C})$$

is an arbitrary smooth matrix-function with determinant 1. Why determinant 1? Because I want to preserve the metric.

Principal and subprincipal symbols transform as

$$L_{\mathrm{prin}} \mapsto R^* L_{\mathrm{prin}} R,$$

$$L_{\mathrm{sub}} \mapsto R^* L_{\mathrm{sub}} R + \frac{i}{2}\left(R^*_{x^\alpha} (L_{\mathrm{prin}})_{p_\alpha} R - R^* (L_{\mathrm{prin}})_{p_\alpha} R_{x^\alpha} \right).$$

Problem: subprincipal symbol does not transform covariantly.

Solution: define *covariant* subprincipal symbol $L_{\mathrm{csub}}(x)$ as

$$L_{\mathrm{csub}} := L_{\mathrm{sub}} + \frac{i}{16}\, g_{\alpha\beta}\{L_{\mathrm{prin}}, \operatorname{adj} L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}\,,$$

where

$$\{U,V,W\} := U_{x^\alpha} V W_{p_\alpha} - U_{p_\alpha} V W_{x^\alpha}$$

is the generalised Poisson bracket on matrix-functions and adj is the operator of matrix adjugation

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \operatorname{adj} U$$

from elementary linear algebra.

Electromagnetic covector potential appears out of thin air

Fact: covariant subprincipal symbol can be rewritten as

$$L_{\mathrm{csub}}(x) = L_{\mathrm{prin}}(x, A(x)), \tag{1}$$

where A is a real-valued covector field.

Explanation: the matrices $(L_{\mathrm{prin}})_{p_\alpha}$, $\alpha = 1, 2, 3, 4$, are Pauli matrices and these form a basis in the real vector space of 2×2 Hermitian matrices. Formula (1) is simply an expansion of the matrix L_{csub} with respect to the basis of Pauli matrices.

Definition The adjugate of a 2×2 matrix differential operator L is an operator whose principal and covariant subprincipal symbols are matrix adjugates of those of the original operator L.

I denote matrix adjugation adj and operator adjugation Adj .

Non-geometric representation of the Dirac equation

Theorem [J. Phys. A: Math. Theor. **48** (2015) 165203] The Dirac equation in curved spacetime can be written as a system of 4 equations

$$\begin{pmatrix} L & mI \\ mI & \mathrm{Adj}\,L \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 0.$$

Here m is the electron mass, I is the 2×2 identity matrix, and v and w are unknown 2-columns of complex-valued scalar fields.

Four fundamental equations of theoretical physics

1 Maxwell's equations. Describe electromagnetism and photons.

2 Dirac equation. Describes electrons and positrons.

3 Massless Dirac equation. Describes* neutrinos and antineutrinos.

4 Linearized Einstein field equations of general relativity. Describe gravity.

All four contain the same physical constant, the speed of light.

*OK, I know that neutrinos actually have a small mass.

Accepted explanation: theory of relativity

God is a geometer. He created a 4-dimensional world parameterized by local coordinates x^1, x^2, x^3, x^4 (here x^4 is time), in which distances are measured in a funny, Lorentzian, way.

Having decided to use the Lorentzian metric, God then wrote down the main equations of theoretical physics using **only geometric constructions**, i.e. using concepts such as connection, curvature etc. This way all equations have the same physical constant, the speed of light, encoded in them.

Alternative explanation

God is an analyst. He created a 4-dimensional world, then wrote down a single system of nonlinear PDEs which describes all phenomena in this world. In doing this, God did not have a particular way of measuring distances in mind. This system of PDEs has different solutions which we interpret as electromagnetism, gravity, electrons, neutrinos etc. The reason the same physical constant, the speed of light, manifests itself in all physical phenomena is because we are looking at different solutions of the **same** system of PDEs.

Potential advantage of formulating a field theory in “non-geometric” terms: there might be a chance of describing the interaction of physical fields in a more consistent (non-perturbative?) manner.