Periodic spectral problem for the massless Dirac operator

Michael Levitin (Reading) and Dmitri Vassiliev (UCL)

25 March 2015

Workshop “Periodic and Other Ergodic Problems”

Isaac Newton Institute for Mathematical Sciences, Cambridge

Massless Dirac operator

Journal of the LMS, 2014, vol. 89, p. 301–320.

Let M be a 3-dimensional connected compact oriented manifold without boundary equipped with a Riemannian metric $g_{\alpha\beta}$, $\alpha, \beta = 1, 2, 3$ being the tensor indices. Will denote local coordinates by $x = (x^1, x^2, x^3)$.

Choose a triple of smooth orthonormal vector fields e_j, $j = 1, 2, 3$. This is the *frame*. Each vector $e_j(x)$ has coordinate components ${e_j}^\alpha(x)$, $\alpha = 1, 2, 3$.

The *coframe* e^j, $j = 1, 2, 3$, is defined via the relation

$${e_j}^\alpha {e^k}_\alpha = {\delta_j}^k.$$

Define Pauli matrices

$$\sigma^\alpha(x) := s^j\, e_j{}^\alpha(x)\,, \qquad \sigma_\alpha(x) := s_j\, e^j{}_\alpha(x)\,,$$

where

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s_1\,, \quad s^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = s_2\,, \quad s^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = s_3\,.$$

The massless Dirac operator is the 2×2 matrix operator

$$W := -i\sigma^\alpha \left(\frac{\partial}{\partial x^\alpha} + \frac{1}{4}\sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ {\beta \atop \alpha\gamma} \right\} \sigma^\gamma \right) \right)$$

acting on 2-columns of complex-valued scalars $v : M \to \mathbb{C}^2$.

It is formally self-adjoint with respect to the inner product

$$\int_M w^* v \sqrt{\det g_{\alpha\beta}}\, dx\,.$$

Example: 3-torus equipped with Euclidean metric

We work on the unit torus $\mathbb{T}^3$ parameterized by cyclic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π. Metric is assumed to be Euclidean.

The massless Dirac operator reads

$$W = -i \begin{pmatrix} \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} \end{pmatrix}.$$

Seek eigenfunctions in the form $v(x) = u e^{i m_\alpha x^\alpha}$, $m \in \mathbb{Z}^3$, $u \in \mathbb{C}^2$.

The spectrum is as follows.

- Zero is an eigenvalue of multiplicity two.

- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $\|m\|$ and unique (up to rescaling) eigenfunction of the form $u e^{i m_\alpha x^\alpha}$.

- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $-\|m\|$ and unique (up to rescaling) eigenfunction of the form $u e^{i m_\alpha x^\alpha}$.

Spectral asymmetry

The spectrum is said to be symmetric if, as a set, it is invariant under the transformation $\lambda \to -\lambda$.

M. F. Atiyah, V. K. Patodi and I. M. Singer: there is no reason for the spectrum of a first order system to be symmetric.

The *eta function* of the massless Dirac operator W is defined as

$$\eta_W(s) := \sum \frac{\operatorname{sign}\lambda}{|\lambda|^s},$$

where summation is carried out over all nonzero eigenvalues λ of W, and $s \in \mathbb{C}$ is the independent variable.

Series converges absolutely for $\operatorname{Re} s > 3$. Extends meromorphically to whole s-plane with simple poles. No poles at $s = 3$ and $s = 0$. Quantity $\eta_W(0)$ is called *eta invariant* of the operator W.

Large eigenvalues versus small eigenvalues

Residue of the first pole, at $s = 2$, was calculated in JST, 2013, vol. 3, p. 317-360. Was done for a general first order system.

Unfortunately, for the massless Dirac the residue at $s = 2$ is zero.

J.-M. Bismut and D. S. Freed: for the massless Dirac operator there are also no poles at $s = 1$ and $s = -1$. Eta function is holomorphic in the half-plane $\mathrm{Re}\, s > -2$.

Bottom line: in order to establish spectral asymmetry for the massless Dirac operator should be looking at small eigenvalues.

Subject of this talk

We work on the unit torus $\mathbb{T}^3$ and perturb the metric, $g_{\alpha\beta}(x;\epsilon)$, where ϵ is a small parameter and

$$g_{\alpha\beta}(x;0) = \delta_{\alpha\beta}.$$

Our goal is to derive an asymptotic expansion for the eigenvalue with smallest modulus, $\lambda_0(\epsilon)$.

A minor impediment

Inner product depends on ϵ, so Hilbert space depends on ϵ.

Addressing this issue is easy: work with massless Dirac operator on half-densities

$$W_{1/2} := (\det g_{\kappa\lambda})^{1/4}\, W\, (\det g_{\mu\nu})^{-1/4}.$$

Recall that a half-density is a quantity which under changes of local coordinates transforms as the square root of a density.

A major impediment

The massless Dirac operator commutes with the antilinear operator of charge conjugation

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix} =: \mathsf{C}(v).$$

This implies that all its eigenvalues have even multiplicity. In particular, the eigenvalue with smallest modulus, $\lambda_0(\epsilon)$, has multiplicity two.

Had to develop a perturbation theory which accounts for this charge conjugation symmetry.

Technical impediment I

Spectral asymmetry shows up only at the second step of the perturbation process, i.e. $\lambda_0(\epsilon) = c\,\epsilon^2 + O(\epsilon^3)$ as $\epsilon \to 0$.

Technical impediment II

Intermediate formulae are messy and it is hard to write them in a geometrically invariant way.

We overcame this difficulty by using the geometrically invariant explicit formula for the subprincipal symbol of the massless Dirac operator on half-densities. This was derived in Journal of the LMS, 2014, vol. 89, p. 301–320.

Paper on virtues of subprincipal symbol to appear in J Phys A.

In local coordinates our operator reads

$$L = P^\alpha(x)\frac{\partial}{\partial x^\alpha} + Q(x),$$

where $P^\alpha(x)$ and $Q(x)$ are some 2×2 matrix-functions. The full symbol of the operator L is the matrix-function

$$L(x,p) := iP^\alpha(x)\,p_\alpha + Q(x),$$

where $p = (p_1, p_2, p_3)$ is the dual variable (momentum).

We decompose the full symbol into homogeneous components,

$$L_1(x,p) := iP^\alpha(x)\,p_\alpha, \qquad L_0(x) := Q(x),$$

and define the principal and subprincipal symbols as

$$L_{\mathrm{prin}}(x,p) := L_1(x,p),$$

$$L_{\mathrm{sub}}(x) := L_0(x) + \frac{i}{2}(L_{\mathrm{prin}})_{x^\alpha p_\alpha}(x).$$

Main result (JMP, 2013, vol. 54, article 111503)

Theorem 1 We have

$$\lambda_0(\epsilon) = c\,\epsilon^2 + O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0,$$

where the coefficient c is given by the formula

$$c = \frac{i}{16}\,\varepsilon_{\alpha\beta\gamma} \sum_{m\in\mathbb{Z}^3\setminus\{0\}} \left(\delta_{\mu\nu} - \frac{m_\mu m_\nu}{\|m\|^2}\right) m_\alpha\,\widehat{h}_{\beta\mu}(m)\,\overline{\widehat{h}_{\gamma\nu}(m)}\,.$$

Here

$$h_{\alpha\beta}(x) := \left.\frac{\partial g_{\alpha\beta}}{\partial\epsilon}\right|_{\epsilon=0},$$

the hat stands for the Fourier transform and $\varepsilon_{\alpha\beta\gamma}$ is the totally antisymmetric quantity. Repeated tensor indices indicate summation over the values $1, 2, 3$.

Nonlocal (global) nature of our asymptotic coefficient

Put

$$L_{\gamma\nu\beta\mu} := \frac{i\varepsilon_{\alpha\beta\gamma}}{(2\pi)^3} \sum_{m\in\mathbb{Z}^3\setminus\{0\}} \left(\delta_{\mu\nu} - \frac{m_\mu m_\nu}{\|m\|^2}\right) m_\alpha \int_{\mathbb{T}^3} e^{i(x-y)^\alpha m_\alpha} (\ \cdot\ )\, dy\,,$$

$$P_{\gamma\nu\beta\mu} := \frac{1}{4}(L_{\gamma\nu\beta\mu} + L_{\nu\gamma\beta\mu} + L_{\gamma\nu\mu\beta} + L_{\nu\gamma\mu\beta}).$$

This gives us a first order pseudodifferential operator P acting in the vector space of rank two symmetric complex-valued tensor fields, $s_{\beta\mu} \mapsto P_{\gamma\nu\beta\mu} s_{\beta\mu}$. If we equip this vector space with the natural inner product

$$(r,s) := \int_{\mathbb{T}^3} r_{\alpha\beta}\, \overline{s_{\alpha\beta}}\, dx$$

then our formula for c can be rewritten as

$$c = \frac{1}{128\pi^3}(Ph,h).$$

Eta invariant for the 3-torus

Corollary 1 Suppose that the coefficient c in our asymptotic formula $\lambda_0(\epsilon) = c\,\epsilon^2 + O(\epsilon^3)$ is nonzero. Then

$$\lim_{\epsilon\to 0} \eta_{W(\epsilon)}(0) = 2\,\mathrm{sign}\,c\,.$$

Here $W(\epsilon)$ is the massless Dirac operator for the metric $g_{\alpha\beta}(x;\epsilon)$ and $\eta_{W(\epsilon)}(0)$ is the corresponding eta invariant.

Note: proof relies on the fact that the eta function $\eta_{W(\epsilon)}(s)$ is holomorphic in the half-plane $\mathrm{Re}\,s > -2\,$.

Example of quadratic dependence on ϵ

If

$$g_{\alpha\beta}\,dx^\alpha dx^\beta = \left[dx^1\right]^2 + \left[\left(1+\epsilon\left(\cos x^1\right)\right)dx^2 + \epsilon\left(\sin x^1\right)dx^3\right]^2 + \left[\epsilon\left(\sin x^1\right)dx^2 + \left(1-\epsilon\left(\cos x^1\right)\right)dx^3\right]^2$$

then

$$\lambda_0(\epsilon) = -\frac{\epsilon^2}{2(1-\epsilon^2)} = -\frac{\epsilon^2}{2} + O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0.$$

Example of quartic dependence on ϵ

If

$$g_{\alpha\beta}\, dx^\alpha dx^\beta = \left[dx^1 + \epsilon\left(\cos x^1\right)dx^2 + \epsilon\left(\sin x^1\right)dx^3\right]^2 + \left[dx^2\right]^2 + \left[dx^3\right]^2$$

then

$$\lambda_0(\epsilon) = \frac{2\sqrt{1+\epsilon^2} - 2 - \epsilon^2}{4} = -\frac{\epsilon^4}{16} + O(\epsilon^6) \quad \text{as} \quad \varepsilon \to 0.$$

Work in progress, jointly with Y-L Fang: the 3-sphere

For standard metric eigenvalues are

$$\pm\left(k+\frac{1}{2}\right), \qquad k = 1, 2, \ldots,$$

with multiplicity $k(k+1)$.

Our aim is to derive asymptotic expansions

$$\lambda_{+3/2} = \frac{3}{2} + (a_+)\epsilon + (b_+)\epsilon^2 + O(\epsilon^3),$$

$$\lambda_{-3/2} = -\frac{3}{2} - (a_-)\epsilon - (b_-)\epsilon^2 + O(\epsilon^3).$$

We have $a_+ = a_- = -\frac{1}{2}\frac{\partial \ln V(\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}$, where $V(\epsilon) = \int_{\mathbb{S}^3} \sqrt{\det g_{\alpha\beta}}\, dx$.

Generalised Berger sphere

We work in $\mathbb{R}^4$ equipped with Cartesian coordinates (x^1, x^2, x^3, x^4). Consider the following three covector fields

$$e^1{}_\alpha = \begin{pmatrix} x^4 \\ x^3 \\ -x^2 \\ -x^1 \end{pmatrix}, \qquad e^2{}_\alpha = \begin{pmatrix} -x^3 \\ x^4 \\ x^1 \\ -x^2 \end{pmatrix}, \qquad e^3{}_\alpha = \begin{pmatrix} x^2 \\ -x^1 \\ x^4 \\ -x^3 \end{pmatrix}.$$

These covector fields are cotangent to the 3-sphere

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1.$$

We define the rank 2 tensor

$$g_{\alpha\beta} := \sum_{j,k=1}^{3} c_{jk}\, e^j{}_\alpha\, e^k{}_\beta$$

and restrict it to the 3-sphere. Here the c_{jk} are real constants, elements of a positive symmetric 3×3 matrix.