Analysis of first order systems of PDEs on manifolds without boundary

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Playing field

Let $M^{(n)}$ be an n-dimensional manifold without boundary, $n \geq 2$. Will denote local coordinates by $x = (x^1, \ldots, x^n)$.

A half-density is a quantity $M^{(n)} \to \mathbb{C}$ which under changes of local coordinates transforms as the square root of a density.

Will work with m-columns $v : M^{(n)} \to \mathbb{C}^m$ of half-densities.

Inner product $\langle v, w \rangle := \int_{M^{(n)}} w^* v \, dx$, where $dx = dx^1 \ldots dx^n$.

Want to study a formally self-adjoint first order linear differential operator L acting on m-columns of complex-valued half-densities.

Need an invariant analytic description of my differential operator.

In local coordinates my operator reads

$$L = P^\alpha(x)\frac{\partial}{\partial x^\alpha} + Q(x),$$

where $P^\alpha(x)$ and $Q(x)$ are some $m \times m$ matrix-functions.

The principal and subprincipal symbols are defined as

$$L_{\mathsf{prin}}(x,p) := iP^\alpha(x)\,p_\alpha,$$

$$L_{\mathsf{sub}}(x) := Q(x) + \frac{i}{2}(L_{\mathsf{prin}})_{x^\alpha p_\alpha}(x),$$

where $p = (p_1, \ldots, p_n)$ is the dual variable (momentum).

Fact: L_{prin} and L_{sub} are invariantly defined Hermitian matrix-functions on $T^*M^{(n)}$ and $M^{(n)}$ respectively.

Fact: L_{prin} and L_{sub} uniquely determine the operator L.

We say that our operator L is *elliptic* if

$$\det L_{\mathsf{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M^{(n)} \setminus \{0\}, \tag{1}$$

and *non-degenerate* if

$$L_{\mathsf{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M^{(n)} \setminus \{0\}. \tag{2}$$

The ellipticity condition (1) is a standard condition in the analysis of PDEs. Our non-degeneracy condition (2) is less restrictive and will allow us to describe a certain class of hyperbolic operators.

Setting basic elliptic and hyperbolic problems

Suppose that $M^{(n)}$ is compact, L is elliptic and eigenvalues of $L_{\mathsf{prin}}(x,p)$ are simple.

Want to study the spectral problem

$$Lv = \lambda v.$$

Spectrum is discrete but **not** semi-bounded. Eigenvalues λ_k of the operator L accumulate to $+\infty$ **and** $-\infty$.

Also want to study the Cauchy problem

$$w|_{x^{n+1}=0} = v$$

for the hyperbolic system

$$\left(L - i\partial/\partial x^{n+1}\right)w = 0\,.$$

Operator in LHS is automatically non-degenerate on $M^{(n)} \times \mathbb{R}$.

Objects of study

Object 1. The *propagator* $U(x^{n+1})$, i.e. one-parameter family of operators which solves the Cauchy problem for the hyperbolic operator $L - i\partial/\partial x^{n+1}$ on the extended manifold $M^{(n)} \times \mathbb{R}$.

Object 2. The two *counting functions*

$$N_+(\lambda) := \begin{cases} 0 & \text{if} \quad \lambda \le 0, \\ \sum\limits_{0<\lambda_k<\lambda} 1 & \text{if} \quad \lambda > 0, \end{cases} \qquad N_-(\lambda) := \begin{cases} 0 & \text{if} \quad \lambda \le 0, \\ \sum\limits_{-\lambda<\lambda_k<0} 1 & \text{if} \quad \lambda > 0. \end{cases}$$

Object 3. The *eta function* which measures *spectral asymmetry*

$$\eta(s) := \sum |\lambda_k|^{-s} \operatorname{sign} \lambda_k = \int_0^{+\infty} \lambda^{-s} \left(N'_+(\lambda) - N'_-(\lambda)\right) d\lambda\,.$$

Objectives

Objective 1. Construct the propagator explicitly in terms of oscillatory integrals, modulo an integral operator with an infinitely smooth integral kernel. More specifically, want a two-term (with regards to smoothness) explicit formula for the propagator.

Objective 2. Derive two-term asymptotic expansions for the two counting functions

$$N_\pm(\lambda) = a_\pm \lambda^n + b_\pm \lambda^{n-1} + o(\lambda^{n-1})$$

as $\lambda \to +\infty$, where $a_\pm$ and $b_\pm$ are some real constants. More specifically, our objective is to write down explicit formulae for the asymptotic coefficients $a_\pm$ and $b_\pm$.

Stop! Two-term asymptotics require conditions on periodic trajectories. Better work with mollified counting functions

$$(N_\pm * \rho)(\lambda) = a_\pm \lambda^n + b_\pm \lambda^{n-1} + o(\lambda^{n-1}),$$

where $\rho(\lambda)$ is a function from Schwartz space such that $\hat{\rho}(t)$ has small compact support and $\hat{\rho}(t) = 1$ in a neighbourhood of zero.

Objective 3. Determine the residues of the eta function $\eta(s)$. Maybe even determine the *eta invariant* $\eta(0)$.

Warning: doing microlocal analysis for systems is not easy

1 V.Ivrii, 1980, Soviet Math. Doklady.

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5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

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Formula for the first asymptotic coefficient

$$a_+ = a_- = \frac{1}{(2\pi)^n} \sum_j \int\limits_{0<h^{(j)}<1} dx\, dp\,,$$

where the $h^{(j)}(x,p)$ are the positive eigenvalues of $L_{\mathrm{prin}}(x,p)$.

Formula for the second asymptotic coefficient

$$b_+ = -b_- = -\frac{n}{(2\pi)^n} \sum_j \int_{0<h^{(j)}<1} \Big(\overbrace{[v^{(j)}]^* L_{\text{sub}} v^{(j)}}^{\text{obvious term}} \overbrace{- \frac{i}{2}\{[v^{(j)}]^*, L_{\text{prin}} - h^{(j)} I, v^{(j)}\}}^{\text{Safarov's term}}$$

$$+ \frac{i}{n-1} h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} \Big) dx\, dp\,,$$

where the $v^{(j)}(x,p)$ are the eigenvectors of $L_{\text{prin}}(x,p)$ corresponding to the positive eigenvalues $h^{(j)}(x,p)$,

$$\{P, R\} := P_{x^\alpha} R_{p_\alpha} - P_{p_\alpha} R_{x^\alpha}$$

is the Poisson bracket on matrix-functions and

$$\{P, Q, R\} := P_{x^\alpha} Q R_{p_\alpha} - P_{p_\alpha} Q R_{x^\alpha}$$

is its further generalisation.

The U(1) **connection**

Each eigenvector $v^{(j)}(x,p)$ of $L_{\mathrm{prin}}(x,p)$ is defined modulo a gauge transformation

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)},$$

where

$$\phi^{(j)} : T^*M^{(n)} \setminus \{0\} \to \mathbb{R}$$

is an arbitrary smooth real-valued function. There is a connection associated with this gauge degree of freedom, a U(1) connection on the cotangent bundle (similar to electromagnetism).

The U(1) connection has curvature, and this curvature appears in asymptotic formulae for the counting function and propagator.

Why am I confident that my formulae for $b_\pm$ are correct?

Invariance under gauge transformations of the operator

$$L \mapsto R^* L R,$$

where

$$R : M^{(n)} \to \mathsf{U}(m)$$

is an arbitrary smooth unitary matrix-function.

Two by two operators are special

If $m = 2$ then $\det L_{\mathrm{prin}}$ is a quadratic form in momentum

$$\det L_{\mathrm{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta\,.$$

The coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, \ldots, n$, can be interpreted as components of a (contravariant) metric tensor.

Further on we always assume that $m = 2$.

Dimensions 2, 3 and 4 are special

Lemma If $n \geq 5$, then our metric is degenerate, i.e.

$$\det g^{\alpha\beta}(x) = 0, \qquad \forall x \in M^{(n)}.$$

Further on we always assume that $n \leq 4$.

In dimension four the metric can only be Lorentzian

Lemma If $n = 4$ and our operator L is non-degenerate

$$L_{\text{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M^{(n)} \setminus \{0\},$$

then our metric is Lorentzian, i.e. the metric tensor $g^{\alpha\beta}(x)$ has three positive eigenvalues and one negative eigenvalue.

Note: half-densities are now equivalent to scalars. Just multiply or divide by $|\det g_{\alpha\beta}(x)|^{1/4}$.

Extracting more geometry from our differential operator

Let us perform gauge transformations of the operator

$$L \mapsto R^* L R$$

where

$$R : M^{(4)} \to \mathrm{SL}(2,\mathbb{C})$$

is an arbitrary smooth matrix-function with determinant 1. Why determinant 1? Because I want to preserve the metric.

Principal and subprincipal symbols transform as

$$L_{\mathrm{prin}} \mapsto R^* L_{\mathrm{prin}} R,$$

$$L_{\mathrm{sub}} \mapsto R^* L_{\mathrm{sub}} R + \frac{i}{2}\left(R^*_{x^\alpha}(L_{\mathrm{prin}})_{p_\alpha} R - R^*(L_{\mathrm{prin}})_{p_\alpha} R_{x^\alpha}\right).$$

Problem: subprincipal symbol does not transform covariantly.

Solution: define *covariant* subprincipal symbol $L_{\mathsf{csub}}(x)$ as

$$L_{\mathsf{csub}} := L_{\mathsf{sub}} + \frac{i}{16}\, g_{\alpha\beta}\{L_{\mathsf{prin}}, \operatorname{adj} L_{\mathsf{prin}}, L_{\mathsf{prin}}\}_{p_\alpha p_\beta}\,,$$

where adj stands for the operator of matrix adjugation

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \operatorname{adj} P$$

from elementary linear algebra.

Electromagnetic covector potential appears out of thin air

Fact: covariant subprincipal symbol can be rewritten as

$$L_{\mathsf{csub}}(x) = L_{\mathsf{prin}}(x, A(x)), \tag{3}$$

where A is a real-valued covector field.

Explanation: the matrices $(L_{\mathsf{prin}})_{p_\alpha}$, $\alpha = 1, 2, 3, 4$, are Pauli matrices and these form a basis in the real vector space of 2×2 Hermitian matrices. Formula (3) is simply an expansion of the matrix L_{csub} with respect to the basis of Pauli matrices.

Definition The adjugate of a 2×2 matrix differential operator L is an operator whose principal and covariant subprincipal symbols are matrix adjugates of those of the original operator L.

I denote matrix adjugation adj and operator adjugation Adj .

Non-geometric representation of the Dirac equation in 4D

Theorem The Dirac equation in curved Lorentzian spacetime can be written as a system of 4 equations

$$\begin{pmatrix} L & mI \\ mI & \operatorname{Adj} L \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 0.$$

Here m is the electron mass, I is the 2×2 identity matrix, and v and w are unknown 2-columns of complex-valued half-densities.

Analysis of the 3-dimensional case

Continue studying a 2×2 operator but assume now that $n = 3$ and that the principal symbol is trace-free. Also assume ellipticity, which in this case is equivalent to nondegeneracy.

Lemma Under the above assumptions our metric is Riemannian, i.e. the metric tensor $g^{\alpha\beta}(x)$ is positive definite.

Geometric meaning of asymptotic coefficients in 3D

Theorem

$$a_\pm = \frac{1}{6\pi^2} \int_{M^{(3)}} \sqrt{\det g_{\alpha\beta}}\; dx \,,$$

$$b_\pm = \mp \frac{1}{4\pi^2} \int_{M^{(3)}} (\operatorname{tr} L_{\text{csub}}) \sqrt{\det g_{\alpha\beta}}\; dx \,.$$

If we consider the hyperbolic operator $L - i\partial/\partial x^4$ on the extended manifold $M^{(3)} \times \mathbb{R}$ and express its covariant subprincipal symbol via the electromagnetic covector potential A, we get

$$\frac{1}{2} \operatorname{tr} L_{\text{csub}} = A_4 \,.$$

This means that $\frac{1}{2} \operatorname{tr} L_{\text{csub}}$ is the electric potential.

Two special operators on a Riemannian 3-manifold: massless Dirac operator and the operator curl

- Massless Dirac is a 2×2 operator.

- Geometers drop the adjective "massless".

- "Massless Dirac" $\neq$ "Dirac type".

- Massless Dirac is determined by metric* modulo gauge transformations. There is no electromagnetic field in massless Dirac.

- Massless Dirac commutes with operator of charge conjugation

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix}.$$

All eigenvalues have even multiplicity.

**And spin structure.*

• For massless Dirac the first five asymptotic coefficients of $(N'_+ * \rho)(\lambda)$ and $(N'_- * \rho)(\lambda)$ appear to be the same. Difficult to observe spectral symmetry for large λ.

• Curl is a 3×3 operator.

• Curl is not elliptic.

• Hardly any literature on the spectral theory of operator curl.

• Eigenvalue problem for the Maxwell system reduces to an eigenvalue problem for the operator curl.

• The massless Dirac operator is a mathematical model for the most basic fermion, the neutrino, whereas the operator curl is a mathematical model for the most basic boson, the photon.

Some results for the massless Dirac

- Second asymptotic coefficient for $N_+(\lambda)$ and $N_-(\lambda)$ is zero.

- Might soon write down third asymptotic coefficient $c_+ = c_-$ and difference of sixth asymptotic coefficients $f_+ - f_-$.

- Asymptotic formulae for eigenvalues with smallest modulus. Work on $\mathbb{T}^3$ or $\mathbb{S}^3$ and perturb metric starting from standard one: $g_{\alpha\beta}(x;\epsilon)$, with ϵ being a small parameter. Here asymptotic coefficients are **not** expressed via differential geometric invariants.

- Special families of nontrivial metrics for which eigenvalues can be evaluated explicitly. For $\mathbb{S}^3$ these are generalised Berger spheres. For $\mathbb{T}^3$ there is no name for these special metrics.

Generalised Berger sphere

We work in $\mathbb{R}^4$ equipped with Cartesian coordinates (x^1, x^2, x^3, x^4). Consider the following three covector fields

$$e^1{}_\alpha = \begin{pmatrix} x^4 \\ x^3 \\ -x^2 \\ -x^1 \end{pmatrix}, \qquad e^2{}_\alpha = \begin{pmatrix} -x^3 \\ x^4 \\ x^1 \\ -x^2 \end{pmatrix}, \qquad e^3{}_\alpha = \begin{pmatrix} x^2 \\ -x^1 \\ x^4 \\ -x^3 \end{pmatrix}.$$

These covector fields are cotangent to the 3-sphere

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1.$$

We define the rank 2 tensor

$$g_{\alpha\beta} := \sum_{j,k=1}^{3} c_{jk}\, e^j{}_\alpha\, e^k{}_\beta$$

and restrict it to the 3-sphere. Here the c_{jk} are real constants, elements of a positive symmetric 3×3 matrix.

Two tricks for tackling the operator curl

Making curl elliptic. Introduce a new unknown, a scalar field (pressure), and consider the extended operator

$$\begin{pmatrix} \mathrm{curl} & -\mathrm{grad} \\ \mathrm{div} & 0 \end{pmatrix}. \tag{4}$$

This gives additional eigenvalues, those of the operators $\pm\sqrt{-\Delta}$.

Dealing with double eigenvalues of the principal symbol. The operator (4) reduces to a pair of massless Dirac operators perturbed by lower order terms. Explicit formula representing a covector field and scalar field as a rank two spinor.

Four fundamental equations of theoretical physics

1 Maxwell's equations. Describe electromagnetism and photons.

2 Dirac equation. Describes electrons and positrons.

3 Massless Dirac equation. Describes* neutrinos and antineutrinos.

4 Linearized Einstein field equations of general relativity. Describe gravity.

All four contain the same physical constant, the speed of light.

*OK, I know that neutrinos actually have a small mass.

Accepted explanation: theory of relativity

God is a geometer. He created a 4-dimensional world parameterized by coordinates x^1, x^2, x^3, x^4 (here x^4 is time), in which distances are measured in a funny way:

$$\text{distance}^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2(dx^4)^2\,,$$

where c is the speed of light.

Without the term $-c^2(dx^4)^2$ this would be Pythagoras' theorem. Funny way of measuring distances is called *Minkowski metric*.

Having decided to use the Minkowski metric, God then wrote down the main equations of theoretical physics using **only geometric constructions**, i.e. using concepts such as connection, curvature etc. This way all equations have the same physical constant, the speed of light, encoded in them.

Alternative explanation

God is an analyst. He created a 4-dimensional world, then wrote down a single system of nonlinear PDEs which describes all phenomena in this world. In doing this, God did not have a particular way of measuring distances in mind. This system of PDEs has different solutions which we interpret as electromagnetism, gravity, electrons, neutrinos etc. The reason the same physical constant, the speed of light, manifests itself in all physical phenomena is because we are looking at different solutions of the **same** system of PDEs.

Potential advantage of formulating a field theory in “non-geometric” terms: there might be a chance of describing the interaction of physical fields in a more consistent (non-perturbative?) manner.