First order systems of partial differential equations on manifolds without boundary

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Playing field

Let $M^{(n)}$ be an n-dimensional manifold without boundary, $n \geq 2$. Will denote local coordinates by $x = (x^1, \ldots, x^n)$.

A half-density is a quantity $M^{(n)} \to \mathbb{C}$ which under changes of local coordinates transforms as the square root of a density.

Will work with m-columns $v : M^{(n)} \to \mathbb{C}^m$ of half-densities.

Inner product $\langle v, w \rangle := \int_{M^{(n)}} w^* v \, dx$, where $dx = dx^1 \ldots dx^n$.

Want to study a formally self-adjoint first order linear differential operator L acting on m-columns of complex-valued half-densities.

In local coordinates our operator reads

$$L = P^\alpha(x)\frac{\partial}{\partial x^\alpha} + Q(x),$$

where $P^\alpha(x)$ and $Q(x)$ are some $m \times m$ matrix-functions. The full symbol of the operator L is the matrix-function

$$L(x,p) := iP^\alpha(x)\,p_\alpha + Q(x),$$

where $p = (p_1 \ldots, p_n)$ is the dual variable (momentum).

We decompose the full symbol into homogeneous components,

$$L_1(x,p) := iP^\alpha(x)\,p_\alpha, \qquad L_0(x) := Q(x),$$

and define the principal and subprincipal symbols as

$$L_{\mathrm{prin}}(x,p) := L_1(x,p),$$

$$L_{\mathrm{sub}}(x) := L_0(x) + \frac{i}{2}(L_{\mathrm{prin}})_{x^\alpha p_\alpha}(x).$$

Fact: L_{prin} and L_{sub} are invariantly defined Hermitian matrix-functions on $T^*M^{(n)}$ and $M^{(n)}$ respectively.

Fact: L_{prin} and L_{sub} uniquely determine the operator L.

We say that our operator L is *elliptic* if

$$\det L_{\mathsf{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M^{(n)} \setminus \{0\}, \tag{1}$$

and *non-degenerate* if

$$L_{\mathsf{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M^{(n)} \setminus \{0\}. \tag{2}$$

The ellipticity condition (1) is a standard condition in the analysis of PDEs. Our non-degeneracy condition (2) is less restrictive and will allow us to describe a certain class of hyperbolic operators.

Suppose that $M^{(n)}$ is compact, L is elliptic and eigenvalues of $L_{\text{prin}}(x,p)$ are simple.

Want to study the spectral problem

$$Lv = \lambda v.$$

Spectrum is discrete but **not** semi-bounded. Eigenvalues λ_k of the operator L accumulate to $+\infty$ **and** $-\infty$.

Also want to study the Cauchy problem

$$w|_{x^{n+1}=0} = v$$

for the hyperbolic system

$$\left(L - i\partial/\partial x^{n+1}\right)w = 0\,.$$

Operator in LHS is automatically non-degenerate on $M^{(n)} \times \mathbb{R}$.

Objects of study

Object 1. The *counting function*

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1\,,$$

i.e. number of eigenvalues between zero and a given positive λ.

Object 2. The *propagator* $U(x^{n+1})$, i.e. one-parameter family of operators which solves the Cauchy problem for the hyperbolic operator $L - i\partial/\partial x^{n+1}$ on the extended manifold $M^{(n)} \times \mathbb{R}$.

Objectives

Objective 1. Derive a two-term asymptotic expansion for the counting function

$$N(\lambda) = a\lambda^n + b\lambda^{n-1} + o(\lambda^{n-1})$$

as $\lambda \to +\infty$, where a and b are some real constants. More specifically, our objective is to write down explicit formulae for the asymptotic coefficients a and b.

Objective 2. Construct the propagator explicitly in terms of oscillatory integrals, modulo an integral operator with an infinitely smooth integral kernel. More specifically, want a two-term (with regards to smoothness) explicit formula for the propagator.

Objectives achieved in O.Chervova, R.J.Downes and D.Vassiliev, *Journal of Spectral Theory*, 2013.

Warning: doing microlocal analysis for systems is not easy

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.

4 V.Ivrii, 1984, Springer Lecture Notes.

5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

6 V.Ivrii, book, 1998, Springer.

7 W.J.Nicoll, PhD thesis, 1998, University of Sussex.

8 I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.

Explicit formula for the second asymptotic coefficient

$$b=-\frac{n}{(2\pi)^n}\sum_j\int_{0<h^{(j)}<1}\Big(\overbrace{[v^{(j)}]^*L_{\text{sub}}v^{(j)}}^{\text{obvious term}}\overbrace{-\frac{i}{2}\{[v^{(j)}]^*,L_{\text{prin}}-h^{(j)},v^{(j)}\}}^{\text{Safarov's term}}$$

$$+\frac{i}{n-1}h^{(j)}\{[v^{(j)}]^*,v^{(j)}\}\Big)dx\,dp\,,$$

$h^{(j)}(x,p)$, and $v^{(j)}(x,p)$ are eigenvalues and eigenvectors of $L_{\text{prin}}(x,p)$,

$$\{P,R\}:=P_{x^\alpha}R_{p_\alpha}-P_{p_\alpha}R_{x^\alpha}$$

is the Poisson bracket on matrix-functions and

$$\{P,Q,R\}:=P_{x^\alpha}QR_{p_\alpha}-P_{p_\alpha}QR_{x^\alpha}$$

is its further generalisation.

The U(1) **connection**

Each eigenvector $v^{(j)}(x,p)$ of $L_{\mathsf{prin}}(x,p)$ is defined modulo a gauge transformation

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)},$$

where

$$\phi^{(j)} : T^*M^{(n)} \setminus \{0\} \to \mathbb{R}$$

is an arbitrary smooth function. There is a connection associated with this gauge degree of freedom, a U(1) connection on the cotangent bundle (similar to electromagnetism).

The U(1) connection has curvature, and this curvature appears in asymptotic formulae for counting function and propagator.

Why am I confident that my formulae are correct?

I analyzed what happens when we perform gauge transformations of the original operator

$$L \mapsto R^* L R,$$

where

$$R : M^{(n)} \to \mathsf{U}(m)$$

is an arbitrary smooth unitary matrix-function.

The spectrum does not change under unitary transformations, so the asymptotic coefficients a and b should not change.

Two by two operators are special

Suppose that $m = 2$. Then the determinant of the principal symbol is a quadratic form in momentum

$$\det L_{\mathrm{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta$$

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, \ldots, n$, can be interpreted as components of a (contravariant) Riemannian metric.

Further on we always assume that $m = 2$.

Dimensions 2, 3 and 4 are special

Lemma If $n \geq 5$, then our metric is degenerate, i.e.

$$\det g^{\alpha\beta}(x) = 0, \qquad \forall x \in M^{(n)}.$$

Further on we always assume that $n \leq 4$.

In dimension four the metric can only be Lorentzian

Lemma If $n = 4$ and our operator L is non-degenerate, then our metric is Lorentzian, i.e. metric tensor $g^{\alpha\beta}(x)$ has three positive eigenvalues and one negative eigenvalue.

Note: in dimension four half-densities are equivalent to scalars. Just multiply or divide by $|\det g_{\alpha\beta}(x)|^{1/4}$.

Extracting more geometry from our differential operator

Let us perform gauge transformations of the original operator

$$L \mapsto R^* L R$$

where

$$R : M^{(4)} \to \mathrm{SL}(2,\mathbb{C})$$

is an arbitrary smooth matrix-function with determinant 1. Why determinant 1? Because I want to preserve the metric.

Principal and subprincipal symbols transform as

$$L_{\mathrm{prin}} \mapsto R^* L_{\mathrm{prin}} R,$$

$$L_{\mathrm{sub}} \mapsto R^* L_{\mathrm{sub}} R + \frac{i}{2}\left(R^*_{x^\alpha}(L_{\mathrm{prin}})_{p_\alpha} R - R^*(L_{\mathrm{prin}})_{p_\alpha} R_{x^\alpha}\right).$$

Problem: subprincipal symbol does not transform covariantly.

Solution: define *covariant* subprincipal symbol $L_{\mathrm{csub}}(x)$ as

$$L_{\mathrm{csub}} := L_{\mathrm{sub}} + \frac{i}{16}\, g_{\alpha\beta}\{L_{\mathrm{prin}}, \operatorname{adj} L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}\,,$$

where adj stands for the operator of matrix adjugation

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \operatorname{adj} P$$

from elementary linear algebra.

Electromagnetic covector potential appears out of thin air

Covariant subprincipal symbol can be rewritten as

$$L_{\text{csub}}(x) = L_{\text{prin}}(x, A(x)),$$

where A is a real-valued covector field.

Above formula is simply an expansion of the matrix L_{csub} with respect to the basis $(L_{\text{prin}})_{p_\alpha}$, $\alpha = 1, 2, 3, 4$, in the real vector space of 2×2 Hermitian matrices.

Definition The adjugate of a 2×2 matrix differential operator L is an operator whose principal and covariant subprincipal symbols are matrix adjugates of those of the original operator L.

I denote matrix adjugation adj and operator adjugation Adj.

Non-geometric representation of Dirac operator in 4D

Theorem (arXiv:1401.3160) The Dirac operator in curved 4-dimensional spacetime can be written as a 4×4 matrix operator

$$\begin{pmatrix} L & mI \\ mI & \mathrm{Adj}\, L \end{pmatrix}.$$

Here m is mass and I is the 2×2 identity matrix.

Analysis of the 3-dimensional case

We continue studying a 2×2 operator but assume now that

$$n = 3, \tag{3}$$

$$\operatorname{tr} L_{\mathrm{prin}}(x,p) = 0, \qquad \forall (x,p) \in T^*M^{(3)}. \tag{4}$$

Also, in the remainder of the talk we assume ellipticity (1). Note that under assumptions (3) and (4) the ellipticity condition (1) is equivalent to the non-degeneracy condition (2).

Lemma Under the above assumptions our metric is Riemannian, i.e. metric tensor $g^{\alpha\beta}(x)$ is positive definite.

Geometric meaning of asymptotic coefficients

Theorem (arXiv:1403.2663)

$$a = \frac{1}{6\pi^2} \int_{M^{(3)}} \sqrt{\det g_{\alpha\beta}}\ dx\,, \tag{5}$$

$$b = -\frac{1}{4\pi^2} \int_{M^{(3)}} (\operatorname{tr} L_{\mathrm{csub}}) \sqrt{\det g_{\alpha\beta}}\ dx\,. \tag{6}$$

If we consider the hyperbolic operator $L - i\partial/\partial x^4$ on the extended manifold $M^{(3)} \times \mathbb{R}$ and express its covariant subprincipal symbol via the electromagnetic covector potential A, we get

$$\frac{1}{2} \operatorname{tr} L_{\mathrm{csub}} = A_4\,.$$

This means that $\frac{1}{2} \operatorname{tr} L_{\mathrm{csub}}$ is the electric potential.

Four fundamental equations of theoretical physics

1 Maxwell's equations. Describe electromagnetism and photons.

2 Dirac equation. Describes electrons and positrons.

3 Massless Dirac equation. Describes* neutrinos and antineutrinos.

4 Linearized Einstein field equations of general relativity. Describe gravity.

All four contain the same physical constant, the speed of light.

*OK, I know that neutrinos actually have a small mass.

Accepted explanation: theory of relativity

God is a geometer. He created a 4-dimensional world parameterized by coordinates x^1, x^2, x^3, x^4 (here x^4 is time), in which distances are measured in a funny way:

$$\text{distance}^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2(dx^4)^2\,,$$

where c is the speed of light.

Without the term $-c^2(dx^4)^2$ this would be Pythagoras' theorem. Funny way of measuring distances is called *Minkowski metric*.

Having decided to use the Minkowski metric, God then wrote down the main equations of theoretical physics using **only geometric constructions**, i.e. using concepts such as connection, curvature etc. This way all equations have the same physical constant, the speed of light, encoded in them.

Alternative explanation

God is an analyst. He created a 4-dimensional world, then wrote down one system of nonlinear PDEs which describes phenomena in this world. In doing this, God did not have a particular way of measuring distances in mind. This system of PDEs has different solutions which we interpret as electromagnetism, gravity, electrons, neutrinos etc. The reason the same physical constant, the speed of light, manifests itself in all physical phenomena is because we are looking at different solutions of the **same** system of PDEs.

Action plan: spend next 100 years performing meticulous microlocal analysis of systems of first order PDEs in dimension four.