Spectral theory of first order elliptic systems

Dmitri Vassiliev
(University College London)

24 May 2013

Conference "Complex Analysis & Dynamical Systems VI"

Nahariya, Israel

Typical problem in my subject area

$$-\Delta v = \lambda v \quad \text{in} \quad M \subset \mathbb{R}^3, \qquad v|_{\partial M} = 0.$$

Finding eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ is difficult, so one introduces the counting function

$$N(\lambda) := \sum_{\lambda_k < \lambda} 1$$

(number of eigenvalues below a given positive λ) and studies the asymptotic behaviour of $N(\lambda)$ as $\lambda \to +\infty$.

Rayleigh–Jeans law (1905):

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + o(\lambda^{3/2}) \quad \text{as} \quad \lambda \to +\infty,$$

where V is the volume of M.

Lord Rayleigh's "proof" of the Rayleigh–Jeans law

Suppose the domain M is a cube with side length a. Then the eigenvalues and eigenfunctions can be calculated explicitly:

$$v(x) = \sin\left(\frac{\pi k_1 x^1}{a}\right) \sin\left(\frac{\pi k_2 x^2}{a}\right) \sin\left(\frac{\pi k_3 x^3}{a}\right),$$

$$\lambda = \frac{\pi^2}{a^2}(k_1^2 + k_2^2 + k_3^2),$$

where $x = (x^1, x^2, x^3)$ are Cartesian coordinates in $\mathbb{R}^3$ and k_1, k_2, k_3 are arbitrary natural numbers.

We see that $N(\lambda)$ is the number of integer lattice points in the positive octant of a ball of radius $\frac{a}{\pi}\sqrt{\lambda}$, so

$$N(\lambda) \approx \frac{1}{8}\left(\frac{4}{3}\pi\left(\frac{a}{\pi}\sqrt{\lambda}\right)^3\right) = \frac{a^3}{6\pi^2}\lambda^{3/2} = \frac{V}{6\pi^2}\lambda^{3/2}.$$

Sir James Jeans' contribution to the Rayleigh–Jeans law:

"It seems to me that Lord Rayleigh has introduced an unnecessary factor 8 by counting negative as well as positive values of his integers".

1910: Lorentz visits Göttingen at Hilbert's invitation and delivers a series of lectures "Old and new problems in physics". Lorentz states the Rayleigh–Jeans law as a mathematical conjecture. Hermann Weyl is in the audience.

1912: Weyl publishes a rigorous proof of Rayleigh–Jeans law. Almost incomprehensible.

Comprehensible proof: in R.Courant and D.Hilbert, *Methods of Mathematical Physics* (1924).

Courant's method. Approximate domain M by a collection of small cubes, setting Dirichlet or Neumann boundary conditions on boundaries of cubes. Setting extra Dirichlet conditions raises the eigenvalues whereas setting extra Neumann conditions lowers the eigenvalues. Remains only to a) choose size of cubes correctly in relation to λ and b) estimate contribution of bits of domain near the boundary (we throw them out).

General statement of the problem. Let M be a compact n-dimensional manifold with boundary ∂M. Consider the spectral problem for an elliptic self-adjoint semi-bounded from below differential operator of even order $2m$:

$$Au = \lambda u \quad \text{on} \quad M, \qquad (B^{(j)}u)\Big|_{\partial M} = 0, \quad j = 1, \ldots, m.$$

Has been proven (by many authors over many years) that

$$N(\lambda) = a\lambda^{n/(2m)} + o(\lambda^{n/(2m)}) \quad \text{as} \quad \lambda \to +\infty$$

where the constant a is written down explicitly.

Weyl's Conjecture (1913): one can do better and prove two-term asymptotic formulae for the counting function. Say, for the case of the Laplacian in 3D with Dirichlet boundary conditions Weyl's Conjecture reads

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} - \frac{S}{16\pi}\lambda + o(\lambda) \quad \text{as} \quad \lambda \to +\infty,$$

where S is the surface area of ∂M. For a general partial differential operator of order $2m$ Weyl's Conjecture reads

$$N(\lambda) = a\lambda^{n/(2m)} + b\lambda^{(n-1)/(2m)} + o(\lambda^{(n-1)/(2m)}) \quad \text{as} \quad \lambda \to +\infty,$$

where the constant b can also be written down explicitly.

For the case of a second order operator Weyl's Conjecture was proved, under certain geometric assumptions on the billiard flow, by Victor Ivrii in 1980.

I proved it for operators of arbitrary order in 1984.

My main research publication:

Yu.Safarov and D.Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, American Mathematical Society, 1997 (hardcover), 1998 (softcover).

"In the reviewer's opinion, this book is indispensable for serious students of spectral asymptotics". Lars Hörmander for the Bulletin of the London Mathematical Society.

Idea of the proof of Weyl's Conjecture

Key word: *microlocal analysis*. L.Hörmander (Fields Medal 1962).

Introduce time t and study the "hyperbolic" equation

$$Aw = \left(i\frac{\partial}{\partial t}\right)^{2m} w$$

subject to the initial condition $w|_{t=0} = v$.

The operator which provides the solution to this "Cauchy problem" is the propagator

$$U(t) := e^{-itA^{1/(2m)}}.$$

Fact: the propagator can be constructed explicitly, modulo an integral operator with smooth kernel, in the form of a *Fourier integral operator*. This is a way of doing the Fourier transform for operators with variable coefficients.

A Fourier integral operator is an oscillatory integral. Similar to Feynman diagrams, the variability of coefficients playing role of perturbation. Unlike Feynman diagrams, 100% rigorous.

Having constructed the propagator, recover information about the spectrum using *Fourier Tauberian theorems*. These allow us to perform the inverse Fourier transform from variable t (time) to variable λ (spectral parameter) using incomplete information, with control of error terms.

Similar to Tauberian theorems used in analytic number theory.

Today: will be studying the spectral problem for a system

$$Av = \lambda v,$$

where A is a first order elliptic self-adjoint $m \times m$ matrix differential operator acting on columns of m complex-valued half-densities v over a connected compact n-dimensional manifold M without boundary. The operator is not semi-bounded.

I define the counting function

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1$$

(number of eigenvalues between zero and a given positive λ) and want to study the asymptotic behaviour of $N(\lambda)$,

$$N(\lambda) = a\lambda^n + b\lambda^{n-1} + o(\lambda^{n-1}) \quad \text{as} \quad \lambda \to +\infty.$$

The concept of a symbol of a differential operator

The first order differential operator A reads

$$A = -iB^\alpha \frac{\partial}{\partial x^\alpha} + C,$$

where $B^\alpha(x)$, $\alpha = 1, \ldots, n$, and $C(x)$ are $m \times m$ matrix-functions.

To get the symbol, replace each $\partial/\partial x^\alpha$ by $i\xi_\alpha$, $\alpha = 1, \ldots, n$:

$$A(x, \xi) = B^\alpha \xi_\alpha + C.$$

Principal symbol $A_{\text{prin}}(x, \xi) := B^\alpha \xi_\alpha$.

Subprincipal symbol $A_{\text{sub}}(x) := C + \frac{i}{2}\frac{\partial B^\alpha}{\partial x^\alpha}$.

The asymptotic coefficients a and b should be expressed via $A_{\text{prin}}(x, \xi)$ and $A_{\text{sub}}(x)$ as some sort of integrals.

Warning: doing microlocal analysis for systems is not easy

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.

4 V.Ivrii, 1984, Springer Lecture Notes.

5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

6 V.Ivrii, book, 1998, Springer.

7 W.J.Nicoll, PhD thesis, 1998, University of Sussex.

8 I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.

9 O.Chervova, R.J.Downes and D.Vassiliev, 2012, preprint `http://arxiv.org/abs/1208.6015`, to appear in JST.

Correct formula for the second asymptotic coefficient

$$b = \text{sum of 3 terms}\,.$$

Term 1: the obvious one, proportional to $A_{\mathsf{sub}}(x)$.

Term 2: Safarov's term. Involves eigenvalues and eigenvectors of $A_{\mathsf{prin}}(x,\xi)$ and their Poisson brackets. Poisson bracket understood in the matrix sense.

Term 3: similar to term 2, only depends on dimension n in a funny way. Contains factor $\frac{1}{n-1}$.

See formula (1.24) in `http://arxiv.org/abs/1208.6015` for details.

Analysis of simplest possible special case

- Our manifold has dimension 3.

- The number of equations in our system is 2.

- The principal symbol is trace-free.

- The subprincipal symbol is zero.

Geometric object 1: the metric

The determinant of the principal symbol is a negative definite quadratic form

$$\det A_{\mathrm{prin}}(x,\xi) = -g^{\alpha\beta}\xi_\alpha\xi_\beta$$

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha,\beta = 1,2,3$, can be interpreted as components of a (contravariant) Riemannian metric.

Geometric object 2: normalised spinor field

Consider the equivalence class of principal symbols with given metric. Choose a representative $\mathbf{A}_{\mathsf{prin}}(x,\xi)$ and use it as a reference. Then any other* principal symbol from this equivalence class can be written as

$$A_{\mathsf{prin}}(x,\xi) = R(x)\,\mathbf{A}_{\mathsf{prin}}(x,\xi)\,R^*(x)\,,$$

where $R: M \to \mathsf{SU}(2)$ is a special unitary matrix-function.

$$R = \begin{pmatrix} \overline{\zeta^1} & \overline{\zeta^2} \\ -\zeta^2 & \zeta^1 \end{pmatrix}, \qquad |\zeta^1|^2 + |\zeta^2|^2 = 1.$$

The pair of complex numbers $\zeta = \begin{pmatrix} \zeta^1 & \zeta^2 \end{pmatrix}$ is called a *spinor*.

* There are some topological issues here. To overcome them, assume, for simplicity, that $A_{\mathsf{prin}}(x,\xi)$ is sufficiently close to $\mathbf{A}_{\mathsf{prin}}(x,\xi)$.

Geometric meaning of the second asymptotic coefficient

$$b = \frac{S(\zeta)}{2\pi^2}\ ,$$

where $S(\zeta)$ is the massless Dirac action with Pauli matrices $\mathbf{B}^\alpha(x)$, $\alpha = 1, 2, 3$, the matrix-functions which appear in the formula for the reference principal symbol $\mathbf{A}_{\mathsf{prin}}(x,\xi) = \mathbf{B}^\alpha \xi_\alpha$.

Bottom line: the differential geometry of spinors is encoded within the microlocal analysis of PDEs.

Four fundamental equations of theoretical physics

1 Maxwell's equations. Describe electromagnetism and photons.

2 Dirac equation. Describes electrons and positrons.

3 Massless Dirac equation. Describes* neutrinos and antineutrinos.

4 Linearized Einstein field equations of general relativity. Describe gravity.

All four contain the same physical constant — the speed of light.

*OK, I know that neutrinos actually have a small mass.

Accepted explanation: theory of relativity

God is a geometer. He created a 4-dimensional world parameterized by coordinates x^0, x^1, x^2, x^3 (here x^0 is time), in which distances are measured in a funny way:

$$\text{distance}^2 = -c^2(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \,,$$

where c is the speed of light.

Without the term $-c^2(dx^0)^2$ this would be Pythagoras' theorem. Funny way of measuring distances is called *Minkowski metric*.

Having decided to use the Minkowski metric, God then wrote down the main equations of theoretical physics using **only geometric constructions**, i.e. using concepts such as connection, curvature etc. This way all equations have the same physical constant — the speed of light — encoded in them.

Alternative explanation

God is an analyst. He created a 3-dimensional Euclidean world, then added (absolute) time and wrote down one system of nonlinear PDEs which describes phenomena in this world. This system of PDEs has different solutions which we interpret as electromagnetism, gravity, electrons, neutrinos etc. The reason the same physical constant — the speed of light — manifests itself in all physical phenomena is because we are looking at different solutions of the **same** system of PDEs.

I cannot write down the unifying system of field equations but I can extract most geometric constructs of theoretical physics from systems of PDEs, simply by performing microlocal analysis.

Action plan: spend next 100 years performing meticulous microlocal analysis of systems of nonlinear hyperbolic PDEs.