Problems in the spectral theory of differential operators

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Basic example of a problem in this subject area: acoustic resonator. Suppose we are studying the vibrations of air

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x_1^2} - \frac{\partial^2\varphi}{\partial x_2^2} - \frac{\partial^2\varphi}{\partial x_3^2} = 0$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ subject to boundary conditions

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0.$$

Here φ is the velocity potential and c is the speed of sound.

Seek solutions in the form $\varphi(t, x_1, x_2, x_3) = e^{-i\omega t}\psi(x_1, x_2, x_3)$ where ω is the unknown natural frequency.

This leads to an eigenvalue problem:

$$-\Delta\psi = \lambda\psi \quad \text{in} \quad \Omega, \qquad \partial\psi/\partial n|_{\partial\Omega} = 0,$$

where $\lambda := \omega^2/c^2$ is the spectral parameter.

Finding eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ is difficult, so one introduces the counting function

$$N(\lambda) := \sum_{0 \leq \lambda_k < \lambda} 1$$

("number of eigenvalues below a given λ") and studies the asymptotic behaviour of $N(\lambda)$ as $\lambda \to +\infty$.

Rayleigh–Jeans law (1905):

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + o(\lambda^{3/2}) \quad \text{as} \quad \lambda \to +\infty$$

where V is the volume of the resonator.

Rayleigh's "proof" of the Rayleigh–Jeans law

Suppose Ω is a cube with side length a. Then the eigenvalues and eigenfunctions can be calculated explicitly:

$$\psi_{\mathbf{k}} = \cos\left(\frac{\pi k_1 x_1}{a}\right)\cos\left(\frac{\pi k_2 x_2}{a}\right)\cos\left(\frac{\pi k_3 x_3}{a}\right),$$

$$\lambda_{\mathbf{k}} = \frac{\pi^2}{a^2}\|\mathbf{k}\|^2 = \frac{\pi^2}{a^2}(k_1^2 + k_2^2 + k_3^2),$$

where $\mathbf{k} = (k_1, k_2, k_3)$ and k_1, k_2, k_3 are nonnegative integers.

$N(\lambda)$ is the number of integer lattice points in the nonnegative octant of a ball of radius $\frac{a}{\pi}\sqrt{\lambda}$, so

$$N(\lambda) \approx \frac{1}{8}\left(\frac{4}{3}\pi\left(\frac{a}{\pi}\sqrt{\lambda}\right)^3\right) = \frac{a^3}{6\pi^2}\lambda^{3/2} = \frac{V}{6\pi^2}\lambda^{3/2}.$$

Rigorous proof of Rayleigh–Jeans law: H.Weyl (1912). Almost incomprehensible.

Comprehensible proof: in R.Courant and D.Hilbert, *Methods of Mathematical Physics* (1924).

Courant's method. Approximate domain Ω by a collection of small cubes, setting Dirichlet or Neumann boundary conditions on boundaries of cubes. Setting extra Dirichlet conditions raises the eigenvalues whereas setting extra Neumann conditions lowers the eigenvalues. Remains only to

- choose size of cubes correctly (in relation to λ) and
- estimate contribution of bits of domain near the boundary (we throw them out).

General statement of the problem. Let M be a compact n-dimensional manifold with boundary ∂M. Consider the spectral problem for an elliptic self-adjoint semi-bounded from below differential operator of even order $2m$:

$$Au = \lambda u \quad \text{on} \quad M, \qquad (B^{(j)}u)\Big|_{\partial M} = 0, \quad j = 1, \ldots, m.$$

Has been proven (by many authors over many years) that

$$N(\lambda) = a\lambda^{n/(2m)} + o(\lambda^{n/(2m)}) \quad \text{as} \quad \lambda \to +\infty$$

where the constant a is written down explicitly.

Weyl's Conjecture (1913): one can do better and prove a two-term asymptotic formula

$$N(\lambda) = a\lambda^{n/(2m)} + b\lambda^{(n-1)/(2m)} + o(\lambda^{(n-1)/(2m)}) \quad \text{as} \quad \lambda \to +\infty$$

where the constant b can also be written down explicitly.

For the case of a second order operator Weyl's Conjecture was proved by V.Ivrii and R.B.Melrose in 1980. I proved it for operators of arbitrary order in 1984.

My main mathematical result:

Theorem Weyl's Conjecture is true if we don't have too many periodic and dead-end billiard trajectories.

Yu.Safarov and D.Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, American Mathematical Society, 1997 (hardcover), 1998 (softcover).

Concept of a principal symbol $A_{2m}(x, \xi)$

Given a partial differential operator A, keep only leading (of order $2m$) derivatives and replace each $\partial/\partial x_k$ by $i\xi_k$, $k = 1, \ldots, n$, to get function $A_{2m}(x, \xi)$. A physicist would call ξ *momentum* and write p instead of ξ.

First asymptotic coefficient $a = (2\pi)^{-n} \displaystyle\int_{A_{2m}(x,\xi)<1} dx\, d\xi$.

Hamiltonian $h(x, \xi) := (A_{2m}(x, \xi))^{1/(2m)}$.

Hamiltonian trajectories

$$\dot{x} = h_\xi(x, \xi), \qquad \dot{\xi} = -h_x(x, \xi).$$

Idea of proof

Key word: *microlocal analysis*. L.Hörmander (Fields Medal 1962).

Introduce time t and study the "hyperbolic" equation

$$Au = \left(i\frac{\partial}{\partial t}\right)^{2m} u.$$

Construct the operator $U(t) := e^{-itA^{1/(2m)}}$. This operator is called the *wave group* (or *unitary group*) and it provides the "solution" to the Cauchy problem (initial value problem) for our "hyperbolic" equation. The wave group can be constructed explicitly, modulo an integral operator with smooth kernel, in the form of a *Fourier integral operator*. This is a way of doing the Fourier transform for operators with variable coefficients.

A Fourier integral operator is an oscillatory integral. Similar to Feynman diagrams, the variability of coefficients playing role of perturbation. Unlike Feynman diagrams, 100% rigorous.

Having constructed the wave group, recover information about the spectrum using *Fourier Tauberian theorems*. These allow us to perform the inverse Fourier transform from variable t (time) to variable λ (spectral parameter) using incomplete information, with control of error terms.

Similar to Tauberian theorems used in number theory.

Appendix *Fourier Tauberian theorems* in our book was written by Michael Levitin.

Example: vibrations of a plate

$$\Delta^2 u = \lambda u \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \qquad u|_{\partial\Omega} = \partial u/\partial n|_{\partial\Omega} = 0.$$

Then

$$N(\lambda) = \frac{S}{4\pi}\lambda^{1/2} + \frac{\beta L}{4\pi}\lambda^{1/4} + o(\lambda^{1/4}) \quad \text{as} \quad \lambda \to +\infty$$

where S is the area of the plate, L is the length of the boundary and

$$\beta = -1 - \frac{\Gamma(3/4)}{\sqrt{\pi}\,\Gamma(5/4)} \approx -1.763.$$

The first asymptotic term was derived by Courant (1922).

Inverting the formula and switching to frequencies $\lambda_N^{1/2}$, we get

$$\lambda_N^{1/2} = \frac{4\pi}{S}N - \frac{2\sqrt{\pi}\,\beta L}{S^{3/2}}\sqrt{N} + o(\sqrt{N}\,) \quad \text{as} \quad N \to +\infty.$$

What I am doing now

Looking at systems

$$Av = \lambda v$$

where A is an elliptic self-adjoint first order $m \times m$ matrix (pseudo)differential operator acting on complex-valued m-columns v over an n-dimensional compact manifold M without boundary. The operator is not necessarily semi-bounded.

Principal symbol $A_1(x,\xi)$ is matrix-valued function on $T^*M \setminus \{0\}$.

The eigenvalues $h^{(j)}(x,\xi)$ of the principal symbol play the role of Hamiltonians.

Warning: doing microlocal analysis for systems is not easy

Main contributors to the spectral theory of systems:

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 V.Ivrii, book, 1984, Springer Lecture Notes.

4 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

5 V.Ivrii, book, 1998, approx 800 pages, Springer.

6 V.Ivrii, future book. 2012? Approx 3000 pages?

7 O.Chervova, R.J.Downes and D.Vassiliev, *The spectral function of a first order system* (in preparation, approx 45 pages).

The U(1) connection

Each eigenvector $v^{(j)}(x,\xi)$ of the principal symbol is defined modulo a gauge transformation

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)}$$

where

$$\phi^{(j)} : T^*M \setminus \{0\} \to \mathbb{R}$$

is an arbitrary smooth function. This gives a U(1) connection characterised by a $2n$-component covector potential

$$i\,(\,[v^{(j)}]^* v^{(j)}_{x^\alpha}\,,\,[v^{(j)}]^* v^{(j)}_{\xi_\beta}\,).$$

Curvature is the exterior derivative of the covector potential.

Scalar curvature of the U(1) connection:

$$-i\{[v^{(j)}]^*, v^{(j)}\}$$

where curly brackets denote the Poisson bracket on matrix-functions

$$\{P, R\} := P_{x^\alpha} R_{\xi_\alpha} - P_{\xi_\alpha} R_{x^\alpha} \,.$$

Correct formula for second asymptotic coefficient:

$$b = b_{\mathsf{Saf}} - \frac{ni}{(2\pi)^n(n-1)} \sum_j \int\limits_{0<h^{(j)}<1} h^{(j)}\{[v^{(j)}]^*, v^{(j)}\}\, dx\, d\xi \,.$$

The teleparallel connection

Consider special case when $m = 2$, $n = 3$, the operator is differential and has trace-free principal symbol.

Define an affine connection as follows. Suppose we have a covector η "based" at the point $y \in M$ and we want to construct a "parallel" covector ξ "based" at the point $x \in M$. This is done by solving the linear system of equations

$$A_1(x, \xi) = A_1(y, \eta).$$

The teleparallel connection has zero curvature and nonzero torsion. It is the opposite of the Levi-Civita connection.

Lemma The scalar curvature of the U(1) connection is expressed via torsion of the teleparallel connection.

The massless Dirac operator

Imagine a single neutrino living in a compact 3-dimensional universe. I want to find the spectrum of energy levels this neutrino can occupy. This leads to an eigenvalue problem

$$Av = \lambda v$$

where A is the massless Dirac operator (a 2×2 matrix first order differential operator) and λ is the spectral parameter (energy).

Note: the explicit formula for the massless Dirac operator in curved space is very complicated.

Now suppose I am a spectral analyst and I want to know whether my differential operator A is a massless Dirac operator, without having to learn the differential geometry of spinors.

Theorem An operator is a massless Dirac operator iff

a) its principal symbol is trace-free,

b) subprincipal symbol is proportional to the identity matrix, and

c) second asymptotic coefficient of the spectral function is zero.

Here *spectral function* is $e(\lambda,x,x) := \sum_{0<\lambda_k<\lambda} \|v_k(x)\|^2$. Counting function and spectral function are related as $N(\lambda) = \int_M e(\lambda,x,x)\,dx$.

Bottom line: the differential geometry of spinors is encoded within the microlocal analysis of PDEs.

More radical approach:

poor man's way of solving nonlinear PDEs

Working on a compact 3-manifold, consider eigenvalue problem

$$Av = \lambda s v$$

where A is a 2×2 matrix first order partial differential operator with trace-free principal symbol and zero subprincipal symbol, and $s = s(x) > 0$ is a scalar weight function. View coefficients as dynamical variables and consider the map

principal symbol of operator and scalar weight $\mapsto$
second asymptotic coefficient of counting function.

Theorem (to be published in a separate paper) The map

principal symbol of operator and scalar weight $\mapsto$
second asymptotic coefficient of counting function.

is equivalent to the map

metric and spinor field $\mapsto$ **static massless Dirac action**.

Explanation

Metric is the determinant of the principal symbol.

A 2×2 trace-free Hermitian matrix is not fully defined by its determinant. The remaining degrees of freedom are called "spinor".

Four fundamental equations of theoretical physics

1 Maxwell's equations. Describe electromagnetism and photons.

2 Dirac equation. Describes electrons and positrons.

3 Massless Dirac equation. Describes* neutrinos and antineutrinos.

4 Linearized Einstein field equations of general relativity. Describe gravity.

All four contain the same physical constant — the speed of light.

*OK, I know that neutrinos actually have a small mass.

Accepted explanation: theory of relativity

God is a geometer. He created a 4-dimensional world parametrized by coordinates x^0, x^1, x^2, x^3 (here x^0 is time), in which distances are measured in a funny way:

$$\text{distance}^2 = -c^2(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

where c is the speed of light.

Without the term $-c^2(dx^0)^2$ this would be Pythagoras' theorem. We do not notice the term $-c^2(dx^0)^2$ in everyday life and think that we live in a 3-dimensional Euclidean world in which Pythagoras' theorem is true*. Funny way of measuring distances is called *Minkowski metric*.

*OK, maybe in a slightly modified version if the world is curved.

Having decided to use the Minkowski metric, God then wrote down the main equations of theoretical physics using **only geometric constructions**. This way all equations have the same physical constant — the speed of light — encoded in them.

Geometric concepts used in modern theoretical physics: metric, U(1) connection, spinor, spin connection, Dirac equation etc. Theoretical physics has become part of differential geometry.

Alternative explanation

God is an analyst. He created a 3-dimensional Euclidean world, then added (absolute) time and wrote down one system of nonlinear PDEs which describes phenomena in this world. This system of PDEs has different solutions which we interpret as electromagnetism, gravity, electrons, neutrinos etc. The reason the same physical constant — the speed of light — manifests itself in all physical phenomena is because we are looking at different solutions of the **same** system of PDEs.

I cannot write down the unifying system of field equations but I can extract the main geometric constructs of theoretical physics from systems of PDEs, simply by performing microlocal analysis.