The spectral function of a first order system

Dmitri Vassiliev
(University College London)

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Will be studying the spectral problem for a system

$$Av = \lambda v$$

where A is an elliptic self-adjoint first order $m \times m$ matrix pseudo-differential operator acting on columns of m complex-valued half-densities v over a connected compact n-dimensional manifold M without boundary. The operator is not necessarily semi-bounded.

Principal symbol $A_1(x, \xi)$ is matrix-valued function on $T^*M \setminus \{0\}$.

The eigenvalues of the principal symbol are denoted $h^{(j)}(x, \xi)$. These $h^{(j)}(x, \xi)$ are assumed to be simple.

Objects of study

Object 1. Wave group (or unitary group)

$$U(t) := e^{-itA} = \sum_k e^{-it\lambda_k} v_k(x) \int_M [v_k(y)]^*(\,\cdot\,)\,dy$$

where λ_k and v_k are eigenvalues and eigenfunctions of operator A.

Object 2. Spectral function

$$e(\lambda,x,x) := \sum_{0<\lambda_k<\lambda} \|v_k(x)\|^2.$$

Object 3. Counting function

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1 = \int_M e(\lambda,x,x)\,dx\,.$$

Objectives

Objective 1: construct wave group explicitly in terms of oscillatory integrals, modulo integral operator with smooth kernel.

Objectives 2 and 3: derive, under appropriate assumptions on Hamiltonian trajectories, two-term asymptotics

$$e(\lambda, x, x) = a(x)\,\lambda^n + b(x)\,\lambda^{n-1} + o(\lambda^{n-1}),$$

$$N(\lambda) = a\lambda^n + b\lambda^{n-1} + o(\lambda^{n-1}),$$

as $\lambda \to +\infty$. Here I expect the real constants a, b and real densities $a(x)$, $b(x)$ to be related as

$$a = \int_M a(x)\,dx, \qquad b = \int_M b(x)\,dx.$$

Warning: doing microlocal analysis for systems is not easy

Main contributors to the spectral theory of systems:

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 V.Ivrii, book, 1984, Springer Lecture Notes.

4 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

5 V.Ivrii, book, 1998, approx 800 pages, Springer.

6 V.Ivrii, future book. 2012? Approx 3000 pages?

7 O.Chervova, R.J.Downes and D.Vassiliev, *The spectral function of a first order system* (in preparation, approx 50 pages).

Safarov's formula for the second asymptotic coefficient

$$b(x) = -n \sum_{j=1}^{m^+} \int_{h^{(j)}(x,\xi)<1} \Big([v^{(j)}]^* A_{\mathsf{sub}} v^{(j)} - \frac{i}{2} \{ [v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)} \} \Big)(x,\xi)\, đ\xi$$

where $v^{(j)}(x,\xi)$ are eigenvectors of principal symbol,

$$\{P,R\} := P_{x^\alpha} R_{\xi_\alpha} - P_{\xi_\alpha} R_{x^\alpha}$$

is the Poisson bracket on matrix-functions and

$$\{P,Q,R\} := P_{x^\alpha} Q R_{\xi_\alpha} - P_{\xi_\alpha} Q R_{x^\alpha}$$

is its further generalisation.

Testing $\mathsf{U}(m)$ invariance of second asymptotic coefficient

Transform the operator as

$$A \mapsto RAR^*$$

where

$$R : M \to \mathsf{U}(m)$$

is an arbitrary smooth unitary matrix-function. The coefficient $b(x)$ should be unitarily invariant.

There's something missing in the integrand of Safarov's formula:

$$[v^{(j)}]^* A_{\mathsf{sub}} v^{(j)} \; - \; \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\} \; + \; ? \; .$$

The U(1) **connection**

Each eigenvector $v^{(j)}(x,\xi)$ of the principal symbol is defined modulo a gauge transformation

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)}$$

where

$$\phi^{(j)} : T^*M \setminus \{0\} \to \mathbb{R}$$

is an arbitrary smooth function. This gives a U(1) connection characterised by a $2n$-component real covector potential (P_α, Q^β) where

$$P_\alpha := i[v^{(j)}]^* v^{(j)}_{x^\alpha}, \qquad Q^\beta := i[v^{(j)}]^* v^{(j)}_{\xi_\beta}.$$

The corresponding covariant derivative is

$$\nabla_\alpha := \partial/\partial x^\alpha + iP_\alpha, \qquad \nabla^\beta := \partial/\partial \xi_\beta + iQ^\beta.$$

This definition ensures that $\nabla v^{(j)}$ is orthogonal to $v^{(j)}$.

The curvature of our U(1) connection is

$$R := i \begin{pmatrix} \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha & \nabla_\alpha \nabla^\delta - \nabla^\delta \nabla_\alpha \\ \nabla^\gamma \nabla_\beta - \nabla_\beta \nabla^\gamma & \nabla^\gamma \nabla^\delta - \nabla^\delta \nabla^\gamma \end{pmatrix}.$$

This curvature is the exterior derivative of the covector potential.

The scalar curvature of our U(1) connection is

$$\nabla_\alpha \nabla^\alpha - \nabla^\alpha \nabla_\alpha = -i\{[v^{(j)}]^*, v^{(j)}\}\,.$$

Recall: curly brackets denote Poisson bracket on matrix-functions

$$\{P, R\} := P_{x^\alpha} R_{\xi_\alpha} - P_{\xi_\alpha} R_{x^\alpha}\,.$$

How curvature of U(1) connection manifests itself in microlocal analysis

$$U(t) \overset{\mathrm{mod}\, C^\infty}{=} \sum_j U^{(j)}(t)$$

where phase function of each oscillatory integral $U^{(j)}(t)$ is associated with corresponding Hamiltonian $h^{(j)}(x,\xi)$. New result:

$$\mathrm{tr}[U^{(j)}(0)]_{\mathrm{sub}} = -i\{[v^{(j)}]^*, v^{(j)}\}\,.$$

Note: $\sum_j \{[v^{(j)}]^*, v^{(j)}\} = 0$.

Correct integrand in formula for second asymptotic coefficient:

$$[v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\} + \frac{i}{n-1} h^{(j)} \{[v^{(j)}]^*, v^{(j)}\}\,.$$

Additional assumption 1: $m = 2$ and $\operatorname{tr} A_1 = 0$

In this case principal symbol has only two eigenvalues, $h^\pm(x,\xi)$, and two eigenvectors, $v^\pm(x,\xi)$. These are related as

$$h^- = -h^+, \qquad v^- = \epsilon\overline{v^+}.$$

where

$$\epsilon := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The antilinear map $v \mapsto \mathsf{C}(v) := \epsilon\overline{v}$ is called *charge conjugation*.

Integrand in formula for second asymptotic coefficient simplifies:

$$[v^+]^* A_{\mathsf{sub}} v^+ \; + \; \frac{n}{n-1} i h^+ \{[v^+]^*, v^+\}\,.$$

Additional assumption 2: the operator A is differential

In this case dimension of manifold can only be $n = 2$ or $n = 3$.

The subprincipal symbol A_{sub} does not depend on the dual variable ξ (momentum) and is a function of x (position) only.

We acquire a geometric object, the metric. The determinant of the principal symbol is a negative definite quadratic form

$$\det A_1(x, \xi) = -g^{\alpha\beta}\xi_\alpha\xi_\beta$$

and coefficients $g^{\alpha\beta}(x)$, $\alpha, \beta = 1, \ldots, n$, can be interpreted as components of a (contravariant) Riemannian metric. Our Hamiltonian (positive eigenvalue of principal symbol) takes the form

$$h^+(x, \xi) = \sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta}$$

and Hamiltonian trajectories become geodesics.

Additional assumption 3: $n = 3$

In this case the manifold M is bound to be parallelizable (and, hence, orientable). Established by examining principal symbol.

Furthermore, we acquire a topological invariant

$$\mathbf{c} := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\,\operatorname{tr}\Big((A_1)_{\xi_1}(A_1)_{\xi_2}(A_1)_{\xi_3}\Big).$$

The number $\mathbf{c}$ can take only two values, $+1$ or -1. It describes the orientation of the principal symbol $A_1(x,\xi)$ relative to the chosen orientation of local coordinates.

Most importantly, we acquire a new differential geometric object, namely, a *teleparallel connection*.

The teleparallel connection

Define an affine connection as follows. Suppose we have a covector η "based" at the point $y \in M$ and we want to construct a "parallel" covector ξ "based" at the point $x \in M$. This is done by solving the linear system of equations

$$A_1(x,\xi) = A_1(y,\eta).$$

The teleparallel connection coefficients $\Gamma^\alpha{}_{\beta\gamma}(x)$ can be written down explicitly in terms of the principal symbol and this allows us to define yet another geometric object — the *torsion tensor*

$$T^\alpha{}_{\beta\gamma} := \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}.$$

The teleparallel connection has zero curvature and nonzero torsion. It is the opposite of the Levi-Civita connection.

More convenient to work with $\overset{*}{T}{}^\alpha{}_\beta := \frac{1}{2}\, T^{\alpha\gamma\delta}\, \varepsilon_{\gamma\delta\beta} \sqrt{\det g_{\mu\nu}}\,.$

Lemma 1 The scalar curvature of the U(1) connection is expressed via torsion of the teleparallel connection as

$$-i\{[v^+]^*, v^+\}(x,\xi) = \frac{\mathbf{c}}{2} \frac{\overset{*}{T}{}^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta}{(g^{\mu\nu}(x)\,\xi_\mu\xi_\nu)^{3/2}}\,.$$

Lemma 1 allows us to evaluate the integral in the formula for the second asymptotic coefficient of the spectral function, giving

$$b(x) = \frac{1}{8\pi^2}\Big(\Big(\mathbf{c}\,\mathrm{tr}\,\overset{*}{T} - 2\,\mathrm{tr}\,A_{\mathrm{sub}}\Big)\sqrt{\det g_{\alpha\beta}}\Big)(x)\,.$$

Note that the two traces appearing in this formula have a different meaning: $\mathrm{tr}\,\overset{*}{T}$ is the trace of a 3×3 tensor whereas $\mathrm{tr}\,A_{\mathrm{sub}}$ is the trace of a 2×2 matrix.

The massless Dirac operator

Imagine a single neutrino living in a compact 3-dimensional universe. I want to find the spectrum of energy levels this neutrino can occupy. This leads to an eigenvalue problem

$$Av = \lambda v$$

where A is the massless Dirac operator, a particular 2×2 matrix first order differential operator with trace-free principal symbol.

Note: the explicit formula for the massless Dirac operator in curved space is very complicated.

Now suppose I am a spectral analyst and I want to know whether my differential operator A is a massless Dirac operator, without having to learn the differential geometry of spinors.

Theorem 1 An operator is a massless Dirac operator iff

a) subprincipal symbol is proportional to the identity matrix, and

b) second asymptotic coefficient of the spectral function is zero.

Note: the conditions of Theorem 1 are invariant under transformations of the operator $A \mapsto RAR^*$ where $R : M \to \mathrm{SU}(2)$.

Note: Theorem 1 does not feel the topological differences between different parallelizations of the manifold M.

Bottom line: the basic differential geometry of spinors is encoded within the microlocal analysis of PDEs.

More radical approach: poor man's way of studying nonlinear PDEs

Working on a compact 3-manifold, consider eigenvalue problem

$$Av = \lambda w v$$

where A is a 2×2 matrix first order partial differential operator with $\operatorname{tr} A_1 = 0$ and $A_{\text{sub}} = 0$, and $w : M \to (0, +\infty)$ is a scalar weight function. Note that this eigenvalue problem is invariant under the gauge transformation

$$A \mapsto fAf, \qquad w \mapsto f^2 w$$

where $f : M \to (0, +\infty)$.

View coefficients as dynamical variables and consider the map

principal symbol of operator and scalar weight $\mapsto$ second asymptotic coefficient of counting function.

Theorem 2 (to be published in a separate paper) The map

principal symbol of operator and scalar weight $\mapsto$
second asymptotic coefficient of counting function.

is equivalent to the map

metric and spinor field $\mapsto$ **static massless Dirac action**.

Explanation

Metric is the determinant of the principal symbol.

A 2×2 trace-free Hermitian matrix is not fully defined by its determinant. The remaining degrees of freedom are called "spinor".

Gauge invariance of the eigenvalue problem $Av=\lambda wv$ manifests itself as the conformal invariance of the massless Dirac action.

Four fundamental equations of theoretical physics

1 Maxwell's equations. Describe electromagnetism and photons.

2 Dirac equation. Describes electrons and positrons.

3 Massless Dirac equation. Describes* neutrinos and antineutrinos.

4 Linearized Einstein field equations of general relativity. Describe gravity.

All four contain the same physical constant — the speed of light.

*OK, I know that neutrinos actually have a small mass.

Accepted explanation: theory of relativity

God is a geometer. He created a 4-dimensional world parametrized by coordinates x^0, x^1, x^2, x^3 (here x^0 is time), in which distances are measured in a funny way:

$$\text{distance}^2 = -c^2(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

where c is the speed of light.

Without the term $-c^2(dx^0)^2$ this would be Pythagoras' theorem. Funny way of measuring distances is called *Minkowski metric*.

Having decided to use the Minkowski metric, God then wrote down the main equations of theoretical physics using **only geometric constructions**, i.e. using concepts such as connection, curvature etc. This way all equations have the same physical constant — the speed of light — encoded in them.

Alternative explanation

God is an analyst. He created a 3-dimensional Euclidean world, then added (absolute) time and wrote down one system of nonlinear PDEs which describes phenomena in this world. This system of PDEs has different solutions which we interpret as electromagnetism, gravity, electrons, neutrinos etc. The reason the same physical constant — the speed of light — manifests itself in all physical phenomena is because we are looking at different solutions of the **same** system of PDEs.

I cannot write down the unifying system of field equations but I can extract some geometric constructs of theoretical physics from systems of PDEs, simply by performing microlocal analysis.

Action plan: spend next 100 years studying systems of nonlinear hyperbolic PDEs, hoping to find soliton-type solutions.